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Abstract. In this paper, we address the problem of weak solutions of Yudovich type for the inviscid MHD equations in two dimensions. The local-in-time existence and uniqueness of these solutions sound to be hard to achieve due to some terms involving Riesz transforms in the vorticity-current formulation. We shall prove that the vortex patches with smooth boundary offer a suitable class of initial data for which the problem can be solved. However this is only done under a geometric constraint by assuming the boundary of the initial vorticity to be frozen in a magnetic field line.

We shall also discuss the stationary patches for the incompressible Euler system (E) and the MHD system. For example, we prove that a stationary simply connected patch with rectifiable boundary for the system (E) is necessarily the characteristic function of a disc.

Contents

1. Introduction

In this paper we shall consider a fluid which is electrically conducting and moves through a prevalent magnetic fields. The interaction between the motion and the magnetic fields are governed by the coupling between Navier-Stokes system and Maxwell's equations in the magnetohydrodynamics

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approximation. The basic equations of hydromagnetics are given by,

$$(1)\qquad \begin{cases} \partial_t v + v\cdot\nabla v - \nu\Delta v + \nabla p = b\cdot\nabla b, & t>0,\, x\in\mathbb{R}^d,\, d\in\{2,3\},\\ \partial_t b + v\cdot\nabla b - \mu\Delta b = b\cdot\nabla v, \\ \text{div } v = \text{div } b = 0, \\ v_{|t=0} = v_0, b_{|t=0} = b_0, \end{cases}$$

where v denotes the velocity of the fluid particles and b the magnetic field which are both assumed to be solenoidal. The pressure p is a scalar function that can be recovered from the velocity and the magnetic field by inverting an elliptic equation. The parameters $\nu, \mu \geq 0$ are called the viscosity and the resistivity, respectively.

The major use of MHD is in liquid metal and plasma physics and the derivation of the governing equations can be done by using Maxwell's equations where we neglect the displacement currents,

$$\text{div } b = 0, \quad \text{curl } b = 4\pi J \quad \text{and} \quad \text{curl } E = -\mu\partial_t b$$

with E the electric field, J the current density and μ the magnetic permeability. To complete the fields equations we need an equation for the current density J which requires some assumption on the nature of the fluid. This is described by Ohm's law

$$J = \sigma\big(E + \mu\, v \times b\big).$$

The combination of the preceding equations will lead to the second equation of the magnetohydrodynamics system (1). When the conducting fluid is in motion currents are induced and the magnetic field will in turn act on the fluid according to Lorenz force $\mathcal{L}$

$$\mathcal{L} \triangleq \mu J \times b = \frac{\mu}{4\pi}\text{curl } b\times b, \quad \text{curl } b\times b = b\cdot\nabla b - \frac{1}{2}\nabla|b|^2.$$

Observe that in the foregoing formula Lorentz force is decomposed into two parts: the first one which appears in the first equation of (1) is called a curvature force and acting toward the center of curvature of the field lines. The second one is a magnetic pressure which acts perpendicular to the magnetic fields and it is implicitly contained in the pressure term p. For a general review about the derivation of the MHD equations and some dynamical aspects of the interaction between the magnetic fields and the velocity we can consult the references [2, 9, 15, 18].

The theoretical study of the MHD system has started with the pioneering work of Alfvén [1] who was the first to describe the generation of electromagnetic-hydrodynamic waves by conducting liquid using the MHD equations. From mathematical point of view a lot of progress has be done from that time. For example, the local well-posedness theory which is a central subject in modern PDEs is carried out in various classical function spaces, see for instance [6, 8, 10, 20, 26, 29, 31, 33, 40, 42] and the references therein. However the global existence of such solutions is an open problem except in two dimensions with the full dissipation $\nu, \mu > 0$.

From now onwards we shall focus only on ideal MHD fluid corresponding to $\nu = \mu = 0$ and therefore the equations become

$$(2)\qquad \begin{cases} \partial_t v + v\cdot\nabla v + \nabla p = b\cdot\nabla b, & x\in\mathbb{R}^d,\, t>0,\\ \partial_t b + v\cdot\nabla b = b\cdot\nabla v \\ \text{div } v = \text{div } b = 0, \\ v_{|t=0} = v_0, b_{|t=0} = b_0, \end{cases}$$

One of the most important consequence of the second equation of (2) and known in the literature by *Alfvén's theorem* is the freezing of the magnetic field lines into the fluid; this means that the magnetic lines follow the motion of the fluid particles. We note that the ideal MHD is quite successful model for large-scale plasma physics and can be illustrated in various phenomena in Earth's magnetosphere and on the sun like the *sunspots*. As we shall see the frozen-in magnetic fields will be of crucial importance in our study of weak solutions of Yudovich type in the two dimensional space.

It is in some extent true that the system (2) is at a formal level a perturbation of the incompressible Euler equations and therefore it is legitimate to see whether the known results for Euler equations work for the MHD system as well. For example, it is proved in [38, 39] that the commutator theory developed by Kato and Ponce in [27] can be successfully implemented leading to the local well-posedness for (2) when the initial data v_0, b_0 belong to the sub-critical Sobolev space $H^s, s > \frac{d}{2}+1$ and the maximal solution satisfies $v, b \in \mathcal{C}([0, T^\star); H^s)$. Whether or not the lifespan $T^\star$ is finite is an outstanding open problem even in two dimensions. However it is well-known that for planar motion and in the absence of the magnetic field $b_0 = 0$, classical solutions are global in time since the vorticity $\omega \triangleq \partial_1 v^2 - \partial_2 v^1$ is transported by the flow, namely we have

$$\partial_t \omega + v \cdot \nabla \omega = 0. \tag{3}$$

This shows that Euler equations have a Hamiltonian structure and gives in turn an infinite family of conservation laws such as $\|\omega(t)\|_{L^p} = \|\omega_0\|_{L^p}$ for any $p \in [1, \infty]$. These global a priori estimates allow Yudovich [43] to relax the classical regularity and establish the global existence and uniqueness only with $\omega_0 \in L^1 \cap L^\infty$. Unfortunately, as we shall see the structure of the vorticity is instantaneously altered for the model (2) due to the effects of the magnetic fields. This fact will be a source of at least two main difficulties. The first one is connected to the global existence of classical solutions where no strong global a priori estimates are known till now. The second one concerns Yudovich solutions whose construction is not at all clear even for short time. This can be clarified through the equations governing the vorticity and the current density $j = \partial_1 b^2 - \partial_2 b^1$,

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = b \cdot \nabla j \\ \partial_t j + v \cdot \nabla j = b \cdot \nabla \omega + 2\partial_1 b \cdot \nabla v^2 - 2\partial_2 b \cdot \nabla v^1. \end{cases} \tag{4}$$

We observe that the magnetic field contributes in the last nonlinear part of the second equation with the quadratic term

$$\mathcal{H}(v, b) \triangleq 2\partial_1 b \cdot \nabla v^2 - 2\partial_2 b \cdot \nabla v^1 \tag{5}$$

which can be described as a linear superposition of the quantities $\mathcal{R}_{ik}\omega\mathcal{R}_{lm}j$, where $\mathcal{R}_{ik} = \partial_i\partial_k\Delta^{-1}$ is the iterated Riesz transform. The main step when we wish to deal with Yudovich solutions is to be able to propagate the $L^p \cap L^\infty$ bound of the vorticity for some finite value of p. This problem is not trivial due to two effects. The first one is the lack of continuity of Riesz transform on the bounded functions; and the second one concerns the nonlinear structure of the term $\mathcal{H}(v, b)$. We point out that even for finite value of p no global a priori estimates are known in the literature and their persistence requires the velocity to be in the Lipschitz class.

One of the main scope of this paper is to be able to construct local unique solutions for a subclass of Yudovich data. In broad terms, we shall see that the vortex patches offer a suitable class of initial data for which the construction of Yudovich solutions is possible. But before stating our result let us briefly discuss what is known for Euler equations with this special initial data. First, we say that a vorticity ω_0 is a patch if it is constant inside a bounded set Ω and vanishes outside, namely and by normalization we can take $\omega_0 = \chi_\Omega$. It is clear from the transport equation (3) that this structure is not altered through the time and the vorticity remains always a patch. This means that for any positive time $\omega(t) = \chi_{\Omega_t}$, with $\Omega_t \triangleq \psi(t, \Omega)$ being the image of Ω by the flow. A connected problem that was raised first in the numerical studies and leading later to a nice theoretical achievement was to understand whether or not the boundary develops finite-time singularities. In [12], Chemin proved that when we start with a smooth boundary, say $\partial\Omega$ belongs to the Hölderian class $C^{1+\varepsilon}, 0 < \varepsilon < 1$, then for any time t the boundary $\partial\Omega_t$ remains in the same class. The basic idea of Chemin is that only the co-normal regularity $\partial_X \omega$ of the vorticity contributes for the Lipschitz norm of the velocity. The choice of the vector fields (X_t) can be done in such a way that it should be tangential to $\partial\Omega_t$ for any positive time. This is satisfied when it is transported by the flow, that is,

$$\partial_t X + v \cdot \nabla X = X \cdot \nabla v. \tag{6}$$

One of the main feature of these vector fields is their commutation with the transport operator $\partial_t + v \cdot \nabla$, which leads to the important equation

$$(\partial_t + v \cdot \nabla)\partial_X \omega = 0. \tag{7}$$

This means that the co-normal regularity of the vorticity is also transported by the flow and this is the crucial tool in the framework of the vortex patches.

Our main concern here is to valid similar results for the MHD equations and as we shall see the situation is slightly more complex. The presence of the magnetic field will contribute with two opposite effects. First, it will destroy the structure of the vortex patches and introduce nonlocal singular operators of Calderón-Zygmund type. Second, the fact that the magnetic field is transported by the flow- it is a push-forward vector field- will be of great importance especially for measuring the co-normal regularity of the vorticity.

Before stating our contribution in this subject we shall discuss a little bit an intermediary problem concerning the stationary patches. This consists in finding simply connected bounded domains Ω and D such that $\omega(t) = \chi_\Omega$ and $j(t) = \chi_D$ define a solution for the vorticity-current formulation (4). We can analyze the same problem for the $2d$ incompressible Euler equations. The only example that we know for this latter model is the Rankine vortices corresponding to the domains with circular shape. We will show that these are in fact the only stationary patches. This expected but non trivial result can be obtained from Fraenkel's theorem on potential theory as we shall see later in Section 5 and whose proof is based on many tools of elliptic equations. For the MHD system, we will conduct the same study with the same tools and our results can be summarized in the following theorem.

Theorem 1.1. *The following assertions hold true.*

I) *Let Ω be a simply connected domain with rectifiable Jordan boundary. Then χ_Ω is a stationary patch for the $2d$ Euler equations if and only if Ω is a ball.*

II) *Let D and Ω be two bounded domains and $\omega_0 = \chi_\Omega$, $j_0 = \chi_D$.*

(1) *If $D = \Omega$ then (ω_0, j_0) is a stationary solution for the MHD system (2).*
(2) *If the boundaries ∂D and $\partial\Omega$ are disjoint and rectifiable then (ω_0, j_0) is a stationary solution for the system (2) if and only if Ω and D are concentric balls.*

Few remarks are in order.

Remark 1.2. *In the statement* **II**$) - (1)$ *of the preceding theorem, there are no constraints on the domain Ω. This is due to the special structure of the inviscid MHD equations: if we take $b_0 = v_0$ then we can readily check that this corresponds to a stationary solution for (2) without pressure. This illustrates one of the deepest and rigid geometric structure of the magnetic field which forces here the motion to be independent in time.*

Remark 1.3. *The stationary patches of Euler equations appear as a special case of rotating patches whose study were done in a series of papers such as [5, 25].*

Some additional remarks and comments will be raised in Section 5. Now we shall come back to the consideration of Yudovich solutions in the framework of vortex patches and we shall formulate a general statement later in Theorem 6.1.

Theorem 1.4. *Let Ω be a simply connected domain of class $W^{2,\infty}$ and $\omega_0 = \chi_\Omega$. Let $b_0 = \nabla^\perp \varphi_0$ be a divergence-free magnetic field such that its current density j_0 belongs to $L^1 \cap W^{1,p}$, with $2 < p < \infty$. Assume that:*

(1) *Compatibility assumption:*

$$b_0 \cdot n = 0 \quad \text{on} \quad \partial\Omega,$$

where n is a normal vector to the boundary $\partial\Omega$.

(2) *There exist two constants $\delta, \eta > 0$ such that*

$$\forall\, x \in \mathbb{R}^2, \quad |\varphi_0(x) - \lambda| < \eta \Longrightarrow |b_0(x)| > \delta \tag{8}$$

where λ is the value of φ_0 on the boundary $\partial\Omega$.

Then there exists $T > 0$ and a unique solution (v, b) for the system (2) with

$$\omega, j \in L^\infty([0, T]; L^1 \cap L^\infty) \quad \text{and} \quad v, b \in L^\infty([0, T]; \mathrm{Lip}).$$

Moreover, for any $t \in [0, T]$, the boundary of $\psi(t, \Omega)$ belongs to $W^{2,\infty}$.

Before giving some details about the proof, we shall give few remarks.

Remark 1.5. *It is worth noting that the compatibility assumption is not only restrictive to the vortex patch problem but appears in the current vortex sheets called also in the literature by the MHD tangential discontinuity. The construction of local in time piecewise smooth solutions apart from a smooth hypersurface Γ_t is known provided that the magnetic field b_0 is tangential to Γ_0 and a stability condition is satisfied at each point of the initial discontinuity. For more details see [13, 34] and the references therein.*

Remark 1.6. *The existence of φ_0 in the foregoing theorem follows from the incompressibility of b_0 which is a Hamiltonian vector field. Moreover, since b_0 is co-normal to the connected curve $\partial\Omega$ according to the assumption (1), then necessarily this curve must be a level set of φ_0 and this justifies the existence of λ in the assumption (2). For more details see Proposition 3.4.*

Remark 1.7. *The compatibility assumption (1) imparts to the magnetic field some rigidity: it must be singular for at least one point inside the domain Ω. This follows easily from the fact that the Hamiltonian φ_0 is constant on the boundary and thus it has a critical point in Ω.*

Remark 1.8. *The condition (8) implies in particular that the extrema of the Hamiltonian function φ_0 should not be located on the regular level surface energy containing the curve $\partial\Omega$. This means somehow that the magnetic field must be regular close to this level set. This assumption is very strong and unfortunately it does not allow to reach Chemin's result for the Euler case corresponding to $b_0 = 0$. It seems that the restriction described by the compatibility assumption (1) is relevant and essential in our analysis since it induces deep algebraic structure; we need that any co-normal vector field to the initial patch must commute with the initial magnetic field. However we can hope to dispense with the non degeneracy assumption of the magnetic field around the boundary which sounds to be a technical artifact.*

Remark 1.9. *As we have already seen, Chemin proved in [12] the global persistence of the $C^{1+\varepsilon}, 0 < \varepsilon < 1$ boundary regularity for the two dimensional Euler equations. But in our main result we require more: the boundary should be at least in the class $W^{2,\infty}$. This is due to the following technical fact: the space $C^{\varepsilon-1} \cap L^\infty$ used naturally to measure the co-normal regularity is not an algebra and to overcome this difficulty we should work with positive index spaces.*

Remark 1.10. *As we shall see next in Lemma 2.4, the assumptions (1) and (2) of Theorem 1.4 are not empty.*

Outline of the proof. The proof uses the standard formalism of vortex patches developed by Chemin in [11, 12] for incompressible Euler equations. As we have already seen, one of the main feature of Euler equations is the commutation of the push-forward vector fields given by (6) with the transport operator leading to the master equation (7). This algebraic property is instantaneously destroyed by the magnetic field which contributes with additional terms as the following equations show

$$(9)\qquad \begin{cases} \partial_t\partial_X\omega + v\cdot\nabla\partial_X\omega = b\cdot\nabla\partial_X j + \partial_{\partial_X b-\partial_b X} j \\ \partial_t\partial_X j + v\cdot\nabla\partial_X j = b\cdot\nabla\partial_X\omega + \partial_{\partial_X b-\partial_b X}\omega + \partial_X\{2\partial_1 b\cdot\nabla v^2 - 2\partial_2 b\cdot\nabla v^1\}. \end{cases}$$

Thus and in order to get similar equations to (4) we should at this stage kill the terms involving the vector field $\partial_X b - \partial_b X$. Therefore we shall assume that the vector fields X and b commute initially and this algebraic property is not altered through the time. For the sake of simplicity we can make the choice $X = b$ and this algebraic constraint will lead in the special case of the vortex patch to the geometric constraint described by the compatibility assumption. It is worth noting that the main obstacle to reach the regularity $C^{1+\varepsilon}$ for the boundary is the estimate of the last term of the system (9) in the space $C^{\varepsilon-1}$ and it is not at all clear how to proceed since $C^{\varepsilon-1}\cap L^\infty$ is not an algebra. Besides the geometric condition stated in the compatibility assumption will force the Hamiltonian magnetic field to be degenerate at least at some points inside the domain Ω and subsequently we shall get from the vortex patch formalism some useful information only far from this singular set. In this region, it is not clear how to construct a non degenerate vector field which commutes with the magnetic field. To circumvent this difficulty we use that the initial data are smooth wherever the magnetic field is degenerating combined with the finite speed of propagation of the transport operator. So for a short time we expect the influence of the singular parts to be localized close to the image by the flow of the initial one. This fact is not quite trivial due to the nonlocal property of Riesz transforms in (5) and thus some elaborated analysis are required. Especially, the truncation of the solutions far from the singular set should be done in a special way by cutting along the streamlines of the magnetic field. We emphasize that in this step we use an algebraic identity for the last term of (4), see (20), combined with Calderón commutator type estimates.

The paper is structured as follows. In Section 2 we recall some classical spaces frequently used in the vortex patch problem. We end this section with some results on the persistence regularity for various transport models. In Section 3 we shall review some basic results on the algebra vector fields . In Section 4 we detail some weak estimates for both the vorticity and the current density. In Section 5 we shall be concerned with the stationary patches and we plan to give the proof of Theorem 1.1. Some general facts on conformal mapping and rectifiable boundaries will be also discussed. Section 6 will be devoted to the proof of Theorem 1.4 and its extension to generalized vortex patches. Finally, we shall close this paper by some commutator estimates.

2. Basic tools

In this section we shall introduce some function spaces and investigate some of their elementary properties. We will also recall few basic results concerning some transport equations. First we need to fix a piece of notation that will be frequently used along this paper.

- For $p\in[1,\infty]$, the space L^p denotes the usual Lebesgue space.
- We denote by C any positive constant that may change from line to line and by C_0 a real positive constant depending on the size of the initial data.
- For any positive real numbers A and B, the notation $A\lesssim B$ means that there exists a positive constant C independent of A and B such that $A\leqslant CB$.
- For any two sets $E, F\subset\mathbb{R}^2$ and $x\in\mathbb{R}^2$, we define

$$d(x,E)\triangleq\inf\{|x-y|; y\in E\};\quad \mathrm{dist}(E,F)\triangleq\inf\{d(x,F); x\in E\}.$$

- For a subset $A\subset\mathbb{R}^2$, we denote by χ_A the characteristic function of A which is defined by

$$\chi_A(x)=\begin{cases} 1, & \text{if}\quad x\in A,\\ 0 & \text{if}\quad x\notin A\end{cases}$$

2.1. **Function spaces.** In what follows we intend to recall the definition of Hölder spaces C^α and Sobolev spaces of type $W^{1,p}$. Let $\alpha \in]0,1[$, we denote by C^α the set of continuous functions $u : \mathbb{R}^d \to \mathbb{R}$ such that

$$\|u\|_{C^\alpha} = \|u\|_{L^\infty} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|^\alpha} < \infty.$$

The Lipschitz class denoted by Lip corresponds to the borderline case $\alpha = 1$,

$$\|u\|_{\mathrm{Lip}} = \|u\|_{L^\infty} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x-y|} < \infty.$$

We shall also make use of the space $C^{1+\alpha}(\mathbb{R}^d)$ which is the set of continuously differentiable functions u such that

$$\|u\|_{C^{1+\alpha}} = \|u\|_{L^\infty} + \|\nabla u\|_{C^\alpha} < \infty.$$

By the same way we can define the spaces $C^{n+\alpha}$, with $n \in \mathbb{N}$ and $\alpha \in]0,1[$.
Now we shall recall Sobolev space $W^{1,p}$ for $p \in [1,\infty]$, which is the set of the tempered distribution $u \in \mathcal{S}'$ equipped with the norm

$$\|u\|_{W^{1,p}} \triangleq \|u\|_{L^p} + \|\nabla u\|_{L^p}.$$

Our next task is to introduce the anisotropic Sobolev spaces, which are the analogous of the anisotropic Hölder spaces introduced by Chemin some years ago in [12].

Definition 2.1. *Let $\varepsilon \in (0,1), p \in [1,\infty]$ and $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth divergence-free vector field. Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a scalar function in $L^1 \cap L^\infty$.*

(1) *We say that u belongs to the space C^ε_X if and only if*

$$\|u\|_{C^\varepsilon_X} \triangleq \|u\|_{L^1\cap L^\infty} + \|\partial_X u\|_{C^{\varepsilon-1}} < \infty.$$

(2) *The function u belongs to the space W^p_X if and only if*

$$\|u\|_{W^p_X} \triangleq \|u\|_{L^1\cap L^\infty} + \|\partial_X u\|_{L^p} < \infty$$

where we denote by

$$\partial_X u = \mathrm{div}(Xu).$$

We will see later in Section 3 some additional properties about the Lie derivative ∂_X.
Now we shall introduce the notion of Hölderian singular support.

Definition 2.2. *Let $\varepsilon \in]0,1[$ and $u : \mathbb{R}^2 \to \mathbb{R}$. We say that $x \notin \Sigma^\varepsilon_{\mathrm{sing}}(u)$ if there exits a smooth function χ defined in a neighborhood of x with $\chi(x) \neq 0$ and χu belongs to C^ε.*
The closed set $\Sigma^\varepsilon_{\mathrm{sing}}(u)$ is called the Hölderian singular support of u of index ε.

Example: Let Ω be a Jordan domain and $u = \chi_\Omega$ be the characteristic function of Ω. Then

$$\Sigma^\varepsilon_{\mathrm{sing}}(u) = \partial\Omega.$$

Moreover, if the boundary is C^1 then

$$\partial_X \chi_\Omega = -(X \cdot \vec{n}) d\sigma_{\partial\Omega},$$

with $d\sigma_{\partial\Omega}$ the arc-length measure on $\partial\Omega$ and $\vec{n}$ the outward unit normal. In the particular case where X is tangential, said also co-normal, to $\partial\Omega$ we get

$$\partial_X \chi_\Omega = 0.$$

Now we shall briefly discuss some elementary results on the Littlewood-Paley theory. First we need to recall the following statement concerning the dyadic partition of the unity.
There exist two radial positive functions $\chi \in \mathcal{D}(\mathbb{R}^d)$ and $\varphi \in \mathcal{D}(\mathbb{R}^d \backslash \{0\})$ such that

i) $\chi(\xi) + \sum_{q\geq 0} \varphi(2^{-q}\xi) = 1; \quad \forall\, q \geq 1,\ \mathrm{supp}\ \chi \cap \mathrm{supp}\ \varphi(2^{-q}) = \varnothing$

ii) $\mathrm{supp}\ \varphi(2^{-j}\cdot) \cap \mathrm{supp}\ \varphi(2^{-k}\cdot) = \varnothing$, if $|j-k| \geq 2$.

For any $v \in \mathcal{S}'(\mathbb{R}^d)$ we set the cut-off operators,

$$\Delta_{-1}v = \chi(\mathrm{D})v\ ;\ \forall q \in \mathbb{N},\ \Delta_q v = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad S_q = \sum_{-1\leq p\leq q-1} \Delta_p.$$

From [4], we split formally the product uv of two distributions into three parts,

$$uv = T_u v + T_v u + R(u,v),$$

with

$$T_u v = \sum_q S_{q-1}u\Delta_q v, \quad R(u,v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \sum_{j=-1}^{1} \Delta_{q+j}.$$

We will make continuous use of Bernstein inequalities (see [12] for instance).

Lemma 2.3. *There exists a constant C such that for $q,k \in \mathbb{N}$, $1 \leq a \leq b$ and for $u \in L^a(\mathbb{R}^d)$,*

$$\begin{aligned} \sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} &\leq C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))}\|S_q u\|_{L^a}, \\ C^{-k}2^{qk}\|\Delta_q u\|_{L^a} &\leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk}\|\Delta_q u\|_{L^a}. \end{aligned}$$

Now we shall recall the characterization of Hölder spaces in terms of the frequency cut-offs. For $s \in [0,\infty[\backslash\mathbb{N}$, the usual norm of C^s is equivalent to

$$\|u\|_{C^s} \approx \sup_{q\geq -1} 2^{qs}\|\Delta_q u\|_{L^\infty}.$$

Now we shall prove that the assumptions of Theorem 1.4 can be satisfied by choosing suitably the magnetic vector field.

Lemma 2.4. *Let Ω be a simply connected domain with boundary in $C^{1+\varepsilon}$ and $\varepsilon \in]0,1[$. Then we can find a Hamiltonian vector field b_0 satisfying the assumptions* (1) *and* (2) *of Theorem 1.4.*

Proof. We will briefly outline the proof of this lemma. First, it is a well-known fact that when the boundary $\partial\Omega$ is at least C^1 then it can be seen as a level set of a smooth function. More precisely, there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}_+$ with the following properties:

$$\partial\Omega = \{x \in \mathbb{R}^2; f(x) = 1\};\ \Omega = f^{-1}(]1,+\infty[);$$

$$\lim_{\|x\|\to\infty} f(x) = 0 \quad \text{and} \quad \nabla f(x) \neq 0, \forall x \in \partial\Omega.$$

For $h > 0$ introduce the sets

$$\Omega_h \triangleq \{x;\ \mathrm{dist}(x,\Omega) \leq h\}; \quad \partial\Omega_h \triangleq \{x;\ \mathrm{dist}(x,\partial\Omega) \leq h\}.$$

Then for $\eta > 0$ sufficiently small, there exists $h > 0$ such that

$$\forall x \in \mathbb{R}^2, \quad |f(x)-1| \leq \eta \Longrightarrow x \in \partial\Omega_h. \tag{10}$$

Now let $\chi : \mathbb{R}^2 \to [0,1]$ be a smooth compactly supported function such that $\chi(x) = 1, \forall x \in \Omega_h$. Set

$$\varphi_0(x) = \chi(x)f(x) \quad \text{and} \quad b_0 = \nabla^\perp\varphi_0.$$

Then b_0 satisfies the assumptions (1) and (2) of Theorem 1.4. Indeed, the first assumption is easy to check. As to the second one, using (10) we easily obtain

$$\{|\chi f - 1| \leq \eta\} \subset \partial\Omega_h.$$

Moreover, it is clear that for $x \in \Omega_h$,

$$b_0(x) = \chi(x)\nabla^\perp f(x) = \nabla^\perp f(x).$$

Since $\partial\Omega$ is a regular energy curve, then we can choose $h > 0$ small enough such that, for some $\delta > 0$,

$$\forall x \in \partial\Omega_h; \quad |b_0(x)| \geq \delta.$$

This concludes the proof of the lemma. □

2.2. **Transport equations.** We intend to discuss some basic results about the persistence regularity for some transport equations. The first one is very classical and whose proof can found in [12] for instance.

Proposition 2.5. *Let v be a divergence-free vector field and F be a smooth function. Let f be a solution of the transport equation*

$$\partial_t f + v \cdot \nabla f = F.$$

Then the following estimates hold true.

(1) *L^p-estimates: Let $p \in [1,\infty]$ then for any $t \geq 0$*

$$\|f(t)\|_{L^p} \leq \|f(0)\|_{L^p} + \int_0^t \|F(\tau)\|_{L^p} d\tau.$$

(2) *Hölder estimates: For $\varepsilon \in]-1,1[$ we get*

$$\|f(t)\|_{C^\varepsilon} \leq Ce^{CV(t)}\Big(\|f(0)\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\|F(\tau)\|_{C^\varepsilon} d\tau\Big),$$

with C a constant depending only on the index regularity ε and

$$V(t) \triangleq \int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau.$$

Next we shall deal with the same problem for a coupled transport model generalizing the previous one and which appears naturally in the structure of the MHD system (2).

$$\begin{cases} \partial_t f + v \cdot \nabla f = b \cdot \nabla g + F \\ \partial_t g + v \cdot \nabla g = b \cdot \nabla f + G \end{cases} \tag{11}$$

where F and G are given and the unknowns are f and g.

Proposition 2.6. *Let v and b be two divergence-free smooth vector fields and f, g be two smooth solutions for (11). Then the following estimates hold true.*

(1) *L^p-estimates: For $p \in [1,\infty]$ we get*

$$\|f(t)\|_{L^p} + \|g(t)\|_{L^p} \lesssim \|f(0)\|_{L^p} + \|g(0)\|_{L^p} + \int_0^t \big(\|F(\tau)\|_{L^p} + \|G(\tau)\|_{L^p}\big) d\tau$$

(2) *Hölder estimates: For $\varepsilon \in]-1,1[$ we get*

$$\|f(t)\|_{C^\varepsilon} + \|g(t)\|_{C^\varepsilon} \leq Ce^{CV(t)}\Big(\|f(0)\|_{C^\varepsilon} + \|g(0)\|_{C^\varepsilon} + \int_0^t e^{-CV(\tau)}\big(\|F(\tau)\|_{C^\varepsilon} + \|G(\tau)\|_{C^\varepsilon}\big) d\tau\Big),$$

with

$$V(t) \triangleq \int_0^t \big(\|\nabla v(\tau)\|_{L^\infty} + \|\nabla b(\tau)\|_{L^\infty}\big) d\tau.$$

Proof. We shall introduce Elasser variables, see [19],

$$\Phi \triangleq f + g \quad \text{and} \quad \Psi \triangleq f - g.$$

Then we can easily check that

$$\begin{cases} \partial_t \Phi + (v - b) \cdot \nabla \Phi = F + G \\ \partial_t \Psi + (v + b) \cdot \nabla \Psi = F - G \end{cases}$$

These are transport equations with divergence-free vector fields and thus we can apply Proposition 2.5 leading to the desired estimates. □

3. Basic results on vector fields

In this section we shall review some general results on vector fields and focus on some canonical commutation relations. Special attention will be paid to the Hamiltonian vector fields for which some nice properties are established. Most of the results that will be discussed soon are very known and for the commodity of the reader we prefer giving the proofs of some of them.

3.1. **Push-forward.** Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a smooth vector field and $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. We denote by $\partial_X f$ the derivative of f in the direction X, that is,

$$X(f) = \partial_X f = \sum_{i=1}^{n} X^i \partial_i f = X \cdot \nabla f.$$

This is the Lie derivative of the function f with respect to the vector field X, denoted usually by $\mathcal{L}_X f$ and in the preceding formula we adopt different notations for this object.
For two vector fields $X, Y : \mathbb{R}^n \to \mathbb{R}^n$, their commutator is given by the Lie bracket $[X, Y]$ defined in the coordinates system by

$$\begin{aligned} [X, Y]^i &\triangleq \sum_{i=1}^{n} \big(X^j \partial_j Y^i - Y^j \partial_j X^i \big) \\ &= \partial_X Y^i - \partial_Y X^i. \end{aligned}$$

This can also be written in the form

$$\partial_X \partial_Y - \partial_Y \partial_X = \partial_{\partial_X Y - \partial_y X}. \tag{12}$$

We mention that when f is not sufficiently smooth, for example $f \in L^\infty$ and this will be mostly the case in our context, and the vector field X is divergence-free we define $\partial_X f$ in a weak sense as follows,

$$\partial_X f = \operatorname{div}(X f).$$

Now we intend to study some geometric and analytic properties of the *push-forward* of a vector field X_0 by the flow map associated to another time-dependent vector field $v(t)$. First recall that the push-forward $\phi_\star X$ of a vector field X by a diffeomorphism ϕ of $\mathbb{R}^d$ is given by

$$(\phi_\star X)(\phi(x)) \triangleq X(x) \cdot \nabla \phi(x).$$

Let $v(t)$ be a smooth vector field acting on $\mathbb{R}^n$ and define its flow map by the differential equation

$$\partial_t \psi(t, x) = v(t, \psi(t, x)), \quad \psi(0, x) = x.$$

It is a classical fact that for v belonging to the Lipschitz class the flow map is a diffeomorphism from $\mathbb{R}^d$ to itself and thus the push-forward of the a vector field X_0 by ψ_t is the vector field (X_t) that can be written in the local coordinates in the form

$$X_t(x) = \big(X_0 \cdot \nabla \psi(t) \big)(\psi^{-1}(t, x)). \tag{13}$$

We can easily check by using this formula that the evolution equation governing X_t is given by the transport equation

$$\partial_t X + v \cdot \nabla X = X \cdot \nabla v. \tag{14}$$

Besides, it is a known fact that for two smooth vector fields over $\mathbb{R}^n$, X and Y and for a diffeomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ we have

$$\phi_\star[X, Y] = [\phi_\star X, \phi_\star Y].$$

In the case where ϕ is given by the flow map ψ_t, the above identity can be easily checked using the dynamical equations. As an immediate consequence we see that if two vector fields commute then their push-forward vector fields will commute as well. For a future use of this property it should be better to state it in the next lemma.

Lemma 3.1. *Let $(X_t)_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ be two smooth vector fields solving the equation (14) with the same velocity v. If $[X_0, Y_0] = 0$, then we get*

$$[X_t, Y_t] = 0, \forall t \geq 0.$$

Next we discuss the commutation between the vector fields given by the equation (14) and the material derivative $D_t \triangleq \partial_t + v \cdot \nabla$ and the proof is straightforward.

Proposition 3.2. *Let X be the push-forward of a smooth vector field X_0 defined by (14). Then X commutes with the transport operator $D_t \triangleq \partial_t + v \cdot \nabla$,*

$$\partial_X D_t - D_t \partial_X = 0.$$

3.2. **Hamiltonian vector fields.** We shall discuss now some special structures of Hamiltonian vector fields in two dimensions. To precise the terminology, we say that a smooth vector fields is Hamiltonian if it is divergence-free and in this case there exists a potential scalar function, called *stream function* or *Hamiltonian function*, $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that

$$X(x) = \nabla^\perp \varphi(x) \triangleq \begin{pmatrix} -\partial_2 \varphi \\ \partial_1 \varphi \end{pmatrix}.$$

Notation: Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous vector field. We denote by $\mathcal{Z}_X$ the set of the zeros of X, that is its *singular* set defined by

$$\mathcal{Z}_X = \big\{x \in \mathbb{R}^n, X(x) = 0\big\}.$$

A point x is said to be *regular* for X when $X(x) \neq 0$. Obviously the singular set is closed and the regular one is open.

Definition 3.3. *Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a C^1 Jordan curve and $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field. We say that X is co-normal or tangential to the curve γ if X is regular on γ and*

$$X(x) \cdot n(x) = 0, \quad \forall x \in \gamma,$$

where $n(x)$ denotes a normal vector to the curve at the point x.

Sometimes we use the vocabulary *streamline* or a *field line* for X to denote a curve obeying to the previous definition. This terminology is justified by the next classical result.

Proposition 3.4. *Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a C^1 Jordan curve and $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 Hamiltonian vector field. Then X is co-normal to the curve γ if and only if the curve is a streamline or a level set for the potential function φ, that is there exists a constant λ such that*

$$\varphi(x) = \lambda, \quad \forall x \in \gamma.$$

In this case the vector field X has at least a singular point inside the domain delimited by the curve γ.

Proof. Denote by $t \in [0,1] \mapsto (x_1(t), x_2(t))$ a parametrization of the curve γ. Then a normal vector to the curve is given by $n = (-x_2'(t), x_1'(t))$. Now X is co-normal to this curve means that for any $t \in [0,1]$

$$\nabla^\perp \varphi(x_1(t), x_2(t)) \cdot (-x_2'(t), x_1'(t)) = 0.$$

The left-side term coincides with $\frac{d}{dt}\varphi(x_1(t), x_2(t))$ and thus the co-normal assumption becomes

$$\frac{d}{dt}\varphi(x_1(t), x_2(t)) = 0, \quad \forall t \in [0,1].$$

This is equivalent to say that φ is constant along the curve γ. $\square$

Our next goal is to give a precise description of the push-forward of a Hamiltonian vector-field X_0 and discuss its *fozen-in* property. In broad terms, the vector fields (X_t) transported by a vector field v according to the equation (14) will remain Hamiltonian and the dynamics of the *stream function* will be simply described by a transport equation. This has a deep connection of the *freezing* of the streamlines of vector fields (X_t) into the fluid motion. This latter property was established for the magnetic field and collectively known as *Alfvén's theorem*.

Lemma 3.5. *Let $\varphi_0 : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and $X_0 = \nabla^\perp \varphi_0$. Then the solution to the equation (14) with initial datum X_0 is given by*

$$X(t,x) = \nabla^\perp \varphi(t,x)$$

with φ the unique solution to the problem

$$D_t \varphi = 0, \quad \varphi(0,x) = \varphi_0(x).$$

Proof. It is straightforward computations that the vector field $x \mapsto \nabla^\perp \varphi(t,x)$ satisfies also the equation (14) and thus by uniqueness of the Cauchy problem we get the desired result. $\square$

4. Vorticity-current formulation

Recall that the vorticity of the velocity v coincides in two dimensions with the scalar function $\omega = \partial_1 v^2 - \partial_2 v^1$ and the current density of the magnetic field b is given by $j = \partial_1 b^2 - \partial_2 b^1$. Applying the curl operator to the first equation of (2) and using the notation $D_t = \partial_t + v \cdot \nabla$ to denote the material derivative we get

$$D_t \omega = b \cdot \nabla j.$$

Remark that we have used the following identity: for any two-dimensional vector field X we have

$$\operatorname{curl}(X \cdot \nabla X) = X \cdot \nabla \operatorname{curl} X + \operatorname{curl} X \operatorname{div} X.$$

Performing similar computations for the second equation of (2) one gets

$$D_t j = b \cdot \nabla \omega + \big(\partial_1 b \cdot \nabla v^2 - \partial_2 b \cdot \nabla v^1\big) - \big(\partial_1 v \cdot \nabla b^2 - \partial_2 v \cdot \nabla b^1\big).$$

By straightforward computations we can easily check that

$$\begin{aligned} \partial_1 b \cdot \nabla v^2 - \partial_2 b \cdot \nabla v^1 &= -\big(\partial_1 v \cdot \nabla b^2 - \partial_2 v \cdot \nabla b^1\big) + \omega \operatorname{div} b + j \operatorname{div} v \\ &= -\big(\partial_1 v \cdot \nabla b^2 - \partial_2 v \cdot \nabla b^1\big). \end{aligned}$$

Consequently the MHD system can be written in terms of the coupled equations on ω and j,

$$\begin{cases} D_t \omega = b \cdot \nabla j \\ D_t j = b \cdot \nabla \omega + 2\partial_1 b \cdot \nabla v^2 - 2\partial_2 b \cdot \nabla v^1. \end{cases} \tag{15}$$

For reasons that will be apparent shortly in the proof of Theorem 1.4 we shall need some algebraic structure especially for the last term of (15).

We shall start with the following identities used in [11, 12] and whose proof are very simple. Let $X=(X_1,X_2)$ be a smooth vector field over $\mathbb{R}^2$, then

$$
\begin{aligned}
|X(x)|^2\partial_{11} &= X_1\partial_X\partial_1 - X_2\partial_X\partial_2 + X_2^2\Delta,\\
|X(x)|^2\partial_{22} &= X_2\partial_X\partial_2 - X_1\partial_X\partial_1 + X_1^2\Delta,\\
|X(x)|^2\partial_{12} &= X_1\partial_X\partial_2 + X_2\partial_X\partial_1 - X_1X_2\Delta.
\end{aligned}
\tag{16}
$$

Applying these identities to $\Delta^{-1}\omega$ and using Biot-Savart law $\Delta v = \nabla^\perp\omega$ we get for any $x\in\mathbb{R}^2$

$$
\begin{aligned}
|X(x)|^2\mathcal{R}_{11}\omega &= X_1\partial_X v^2 + X_2\partial_X v^1 + X_2^2\omega,\\
|X(x)|^2\mathcal{R}_{22}\omega &= -X_2\partial_X v^1 - X_1\partial_X v^2 + X_1^2\omega,\\
|X(x)|^2\mathcal{R}_{12}\omega &= -X_1\partial_X v^1 + X_2\partial_X v^2 - X_1X_2\omega,
\end{aligned}
\tag{17}
$$

where we denote by $\mathcal{R}_{ij}$ the Riesz transform $\partial_{ij}\Delta^{-1}$. Therefore we obtain

$$
|X(x)|^2|\nabla v(x)| \lesssim \|X\|_{L^\infty}\|\partial_X v\|_{L^\infty} + \|X\|_{L^\infty}^2\|\omega\|_{L^\infty}. \tag{18}
$$

Similarly we find

$$
|X(x)|^2|\nabla b(x)| \lesssim \|X\|_{L^\infty}\|\partial_X b\|_{L^\infty} + \|X\|_{L^\infty}^2\|j\|_{L^\infty}. \tag{19}
$$

The following lemma will play a crucial role in the proof of Theorem 1.4.

Lemma 4.1. *For smooth divergence-free vector fields X, b and v we get for $X(x)\neq 0$,*

$$
\begin{aligned}
H(v,b) &\triangleq \partial_1 b\cdot\nabla v^2 - \partial_2 b\cdot\nabla v^1\\
&= \frac{2}{|X|^2}\Big\{\partial_X b^1\,\partial_X v^2 - \partial_X b^2\,\partial_X v^1\Big\}\\
&+ \frac{1}{|X|^2}\Big\{j\,X\cdot\partial_X v - \omega\,X\cdot\partial_X b\Big\}.
\end{aligned}
\tag{20}
$$

The dot $\cdot$ denotes the canonical inner product of $\mathbb{R}^2$.

Proof. According to Biot-Savart laws one has

$$
H(v,b) = \mathcal{R}_{12}\omega\big(\mathcal{R}_{11}j - \mathcal{R}_{22}j\big) - \mathcal{R}_{12}j\big(\mathcal{R}_{11}\omega - \mathcal{R}_{22}\omega\big).
$$

Using (17) we get

$$
\mathcal{R}_{11}j - \mathcal{R}_{22}j = \frac{1}{|X(x)|^2}\Big(2X_1\partial_X b^2 + 2X_2\partial_X b^1 + (X_2^2 - X_1^2)j\Big)
$$

and thus

$$
\begin{aligned}
\mathcal{R}_{12}\omega\big(\mathcal{R}_{11}j - \mathcal{R}_{22}j\big) &= \frac{1}{|X(x)|^4}\Big(-X_1\partial_X v^1 + X_2\partial_X v^2 - X_1X_2\omega\Big)\\
&\times \Big(2X_1\partial_X b^2 + 2X_2\partial_X b^1 + (X_2^2 - X_1^2)j\Big).
\end{aligned}
$$

Similarly we get

$$
\begin{aligned}
\mathcal{R}_{12}j\big(\mathcal{R}_{11}\omega - \mathcal{R}_{22}\omega\big) &= \frac{1}{|X(x)|^4}\Big(-X_1\partial_X b^1 + X_2\partial_X b^2 - X_1X_2 j\Big)\\
&\times \Big(2X_1\partial_X v^2 + 2X_2\partial_X v^1 + (X_2^2 - X_1^2)\omega\Big).
\end{aligned}
$$

Subtracting the preceding identities yields to the desired identity. □

4.1. **Weak estimates.** In what follows we shall investigate some weak estimates for the vorticity and the current density.

Proposition 4.2. *Let (ω, j) be a smooth solution of the system (4) then the following results hold true.*

(1) *For $\omega_0, j_0 \in L^p$ with $1 < p < \infty$ we get*
$$\|(\omega, j)(t)\|_{L^p} \le C\|(\omega_0, j_0)\|_{L^p} e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}.$$
(2) *For $\omega_0, j_0 \in L^\infty$ we get*
$$\|(\omega, j)(t)\|_{L^\infty} \le C\|(\omega_0, j_0)\|_{L^\infty} + \int_0^t \|\nabla v(\tau)\|_{L^\infty} \|\nabla b(\tau)\|_{L^\infty} d\tau.$$
(3) *Let $\omega_0, j_0 \in L^1 \cap L^2$, we get*
$$\|(\omega, j)(t)\|_{L^1} \le C\|(\omega_0, j_0)\|_{L^1} + C\|(\omega_0, j_0)\|_{L^2}^2 t\, e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}.$$

Proof. (**1**) Applying Proposition 2.6 to the equation (15) we get
$$\|\omega(t)\|_{L^p} + \|j(t)\|_{L^p} \lesssim \|\omega_0\|_{L^p} + \|j_0\|_{L^p} + \int_0^t \|\nabla b(\tau)\|_{L^p} \|\nabla v(\tau)\|_{L^\infty} d\tau$$
Using the continuity of Riesz transform on L^p with $p \in]1, \infty[$ one gets
$$\|\nabla b(t)\|_{L^p} \lesssim \|j(t)\|_{L^p}$$
which yields in turn
$$\|\omega(t)\|_{L^p} + \|j(t)\|_{L^p} \lesssim \|\omega_0\|_{L^p} + \|j_0\|_{L^p} + \int_0^t \|j(\tau)\|_{L^p} \|\nabla v(\tau)\|_{L^\infty} d\tau.$$
It suffices now to apply Gronwall inequality in order to get the suitable estimate.

(**2**) Using once again Proposition 2.6 implies
$$\|\omega(t)\|_{L^\infty} + \|j(t)\|_{L^\infty} \lesssim \|\omega_0\|_{L^\infty} + \|j_0\|_{L^\infty} + \int_0^t \|\nabla b(\tau)\|_{L^\infty} \|\nabla v(\tau)\|_{L^\infty} d\tau.$$
which is the desired result.

(**3**) Arguing as before and using Hölder inequality we obtain
$$\|\omega(t)\|_{L^1} + \|j(t)\|_{L^1} \lesssim \|\omega_0\|_{L^1} + \|j_0\|_{L^1} + \int_0^t \|\omega(\tau)\|_{L^2} \|j(\tau)\|_{L^2} d\tau$$
At this stage we combine this estimate with the one of the first part (1). ☐

5. Stationary patches

As we can readily observe from the vorticity-current formulation (4) the structure of the initial patches $\omega_0 = \chi_\Omega, j_0 = \chi_D$ cannot be in general conserved in time in contrast with the incompressible Euler equations. This is due peculiarly to the last two terms in the second equation involving Riesz transforms. In what follows we shall look for stationary solutions for (2) in the framework of vortex patches. In other words, we shall characterize the simply connected bounded domains Ω and D such that $\omega(t) = \chi_\Omega$ and $j(t) = \chi_D$ is a solution for the system (4). First observe that when the domains are concentric balls then according to the symmetry invariance of the equations we obtain a stationary solution. We will see that in the case of the disjoint patches these are the only examples of stationary solutions. The proof that we shall present of this intuitive result is not trivial but it will make appeal to a deep result of potential theory which characterize the circle with Newtonian potential. Our result which was introduced in Theorem 1.1 will be now restated only for the inviscid MHD system.

Theorem 5.1. *Let D and Ω be two simply connected domains and $\omega_0 = \chi_\Omega$, $j_0 = \chi_D$. Then the following holds true:*

1. *If $D = \Omega$ then (ω_0, j_0) is a stationary solution for the MHD system (2).*
2. *If the boundaries ∂D and $\partial\Omega$ are disjoint rectifiable Jordan curves then (ω_0, j_0) is a stationary solution for the MHD system (2) if and only if Ω and D are concentric balls.*

Some remarks are in order.

Remark 5.2. *We can deduce from the proof that for the two-dimensional incompressible Euler equations the stationary patches with rectifiable Jordan boundaries are given by χ_Ω with Ω a ball.*

Remark 5.3. *In the case of the 2d incompressible Euler equations we know according to Yudovich result that the patches give rise to unique global solutions. Whether or not the same claim remains true for the MHD equations even locally in time is not at all clear. We will see later in the next section that this can be proved for example for the patches with sufficiently smooth boundaries.*

The proof of Theorem 5.1 relies on Franekel's result and will be divided into two steps depending on the smoothness of the boundaries. The case of C^1 boundaries is more easier than the rectifiable ones and we shall need for this latter case more sophisticated analysis. Especially we will use the conformal mappings to parametrize the boundaries combined with some interesting properties on their boundary behavior. For the clarity of the proofs it would be better to recall some basic results on conformal mappings and rectifiable Jordan curves that will be substantially used later. This will be the subject of the next section.

5.1. **Conformal mappings.** We shall in the first part fix some notation and concepts. Afterwards we discuss the conformal mapping theorem and some basic results on the boundary behavior of the conformal maps.

A planar curve C is called a *Jordan curve* if it is simple and closed meaning that it can be parametrized by an injective continuous function $\gamma : \mathbb{T} \to \mathbb{R}^2$. This curve is said to be *rectifiable* if it is of *bounded variation* and its length L is the total variation of γ. This means that

$$L \triangleq \sup_{(\xi_i)_{i=1}^n \in \mathcal{P}} \sum_{k=1}^{n} |\gamma(\xi_{k+1}) - \gamma(\xi_k)| < \infty$$

where the supremum is taken over all the partition $\mathcal{P}$ of the unit circle $\mathbb{T}$.

The following result due to Riemann is one of the most important results in complex analysis. To restate this result we shall recall the definition of *simply connected* domains. Let $\widehat{\mathbb{C}} \triangleq \mathbb{C} \cup \{\infty\}$ denote the Riemann sphere. We say that a domain $\Omega \subset \widehat{\mathbb{C}}$ is *simply connected* if the set $\widehat{\mathbb{C}} \backslash \Omega$ is connected.

Riemann Mapping Theorem. Let $\mathbb{D}$ denote the unit open ball, $\Omega \subset \mathbb{C}$ be a simply connected domain different from $\mathbb{C}$ and $z_0 \in \Omega$. Then there is a unique bi-holomorphic map (conformal map) $\Phi : \mathbb{D} \to \Omega$ such that

$$\Phi(0) = z_0 \quad \text{and} \quad \arg \Phi'(0) > 0.$$

The area of the domain Ω is given by

$$|\Omega| = \int_{\mathbb{D}} |\Phi'(z)| dA(z),$$

where dA denotes the Lebesgue measure of the plane. In this theorem the regularity of the boundary has no effect regarding the existence of the conformal mapping but as it was shown in various papers it will contribute in the boundary behavior of the conformal mapping, see for instance [35, 41]. One of the main result in this subject dealing with the continuous extension to the boundary goes back to Carathéodory.

Carathéodory Theorem. The conformal map $\Phi : \mathbb{D} \to \Omega$ has one-to-one continuous extension to the closure $\overline{\mathbb{D}}$ if and only if the boundary $\partial\Omega$ is a Jordan curve.

In the next theorem we shall discuss the characterization of rectifiable Jordan curves in terms of the regularity of the associated conformal map. This will require the use of Hardy space of type H^1 which is defined as follows. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function, we define the integral means

$$M(r, f) \triangleq \int_0^{2p} |f(re^{i\theta})| d\theta, \, 0 < r < 1.$$

The function f is said to be of class H^1 if

$$\sup_{0<r<1} M(r, f) < \infty.$$

A classical result known by the name *Hardy's convexity theorem* asserts that $r \mapsto M(r, f)$ is a nondecreasing function and $r \mapsto \log M(r, f)$ is a convex function of $\log r$. For the proof of this result see for example Theorem 1.5 of [17], a reference which provides additional relevant information on the topic.
Next we shall give an analytic characterization of rectifiable curves through the regularity of the conformal mapping.

Theorem 5.4. *The following assertions hold true.*
1) *Let $\Phi : \mathbb{D} \to \Omega$ be the conformal mapping. Then $\partial\Omega$ is rectifiable if and only if $\Phi' \in H^1$ and*

$$L = \lim_{r\to 1} \int_0^{2p} |\Phi'(re^{i\theta})| d\theta.$$

2) *Let $f \in H^1$ then f has an angular limit $f(e^{i\theta})$ almost everywhere on the boundary $\mathbb{T}$ and*

$$\lim_{r\to 1} \int_0^{2p} |f(re^{i\theta})| d\theta = \int_0^{2p} |f(e^{i\theta})| d\theta, \quad \textit{and} \quad \lim_{r\to 1} \int_0^{2p} |f(re^{i\theta}) - f(e^{i\theta})| d\theta = 0.$$

The first result is discussed in Theorem 3.12 of [17]. As to the second one, we refer the reader to Theorem 2.6 of the same reference. An immediate consequence of Theorem 5.4 reads as follows.

Corollary 5.5. *Let Φ be a conformal mapping of the unit ball $\mathbb{D}$ onto the interior of a rectifiable Jordan curve $\partial\Omega$. Then Φ' has an angular limit almost everywhere on the boundary $\mathbb{T}$ and*

$$\lim_{r\to 1} \int_0^{2p} |\Phi'(re^{i\theta})| d\theta = \int_0^{2p} |\Phi'(e^{i\theta})| d\theta, \quad \textit{and} \quad \lim_{r\to 1} \int_0^{2p} |\Phi'(re^{i\theta}) - \Phi'(e^{i\theta})| d\theta = 0.$$

5.2. **Potential characterization of the balls.** There are many results emerging from potential theory with the basic goal to characterize the balls of the Euclidian space $\mathbb{R}^n$. One of them uses the Newtonian potential defined for a domain $\Omega \subset \mathbb{R}^2$ by

$$\varphi(x) = \frac{1}{2p} \int_\Omega \log \frac{1}{|x - y|} dy. \tag{21}$$

In the vocabulary of fluid dynamics this is the stream function of the vorticity χ_Ω. When the domain coincides with a ball then φ is constant on the boundary. The converse is proved by Fraenkel [21], see Theorem 1.1 page 18, that we recall here.

Theorem 5.6. *Let Ω be a bounded domain set of $\mathbb{R}^2$ and φ its Newtonian potential. If φ is constant on the boundary of Ω, then Ω must be a ball.*

The result of Fraenkel is not specific to the two dimensions but can be extended for higher dimensions. Moreover it is worth pointing out that this theorem does not require any assumption on the regularity of the boundary. Recently a partial extension of this result was accomplished by Reichel in [36].

5.3. **Proof of Theorem 5.1.** We intend to give the proof concerning the stationary patches.

Proof. (**1**) This can be deduced from the following fact which is related to the special structure of the inviscid MHD equations (2): if $b_0 = \pm v_0$ then we can easily check that this corresponds to a stationary solution without pressure. This allows to get the desired result.
(**2**) This proof is more tricky and founded on Theorem 5.6. For the sake of clear presentation we shall distinguish smooth boundaries from the rough ones. We mean by *smooth* a C^1 Jordan curve and by *rough* a rectifiable Jordan curve. As we shall see the basic difference between these cases appears when we deal with the flux across the boundary. For smooth boundaries this can be done by using Gauss-Green formula. However for the rough boundaries more sophisticated analysis will be required. To answer to this problem there are at least two approaches that one could consider. The first one is to use a general version of Gauss-Green formula coming from the geometric measure theory. The disadvantage of this formula is that it not so explicit to allow exploitable computations. The second one that will be developed here is to use the conformal mappings. Therefore the problem reduces to measuring the flux across the unit sphere for a modified vector field and by this way we transform the problem into the regularity of the conformal mapping close to the boundary. This has been discussed previously in Section 5.1.

• **Smooth curves.** We assume that the curves $\partial\Omega$ and ∂D are of class C^1, then we may use the classical result concerning the derivative in the distribution sense of the characteristic function χ_D,

$$\nabla\chi_D = -\vec{n}\, d\sigma_{\partial D}, \tag{22}$$

where $d\sigma_{\partial D}$ is the arc-length measure on ∂D and $\vec{n}$ the outward unit normal. Accordingly the first equation of (15) can be written for the stationary patches in the form

$$(v_0 \cdot \vec{n}) d\sigma_{\partial\Omega} = (b_0 \cdot \vec{n}) d\sigma_{\partial D}.$$

Since the boundaries are disjoint then the involved measures are disjointly supported and thus,

$$v_0 \cdot \vec{n} = 0 \quad \text{on} \quad \partial\Omega \qquad \text{and} \qquad b_0 \cdot \vec{n} = 0 \quad \text{on} \quad \partial D. \tag{23}$$

Denote by φ_0 and ψ_0 the stream functions of v_0 and b_0, respectively. They satisfy the elliptic equations

$$\Delta\varphi_0 = \chi_\Omega, \quad \Delta\psi_0 = \chi_D.$$

Now since $v_0 = \nabla^\perp\varphi_0$ and $b_0 = \nabla^\perp\psi_0$ we deduce in view of Proposition 3.4 that the stream functions φ_0 and ψ_0 are constant on the boundaries $\partial\Omega$ and ∂D, respectively. At this stage we can use Fraenkel's theorem and conclude that Ω and D are balls. It remains to show that these balls are concentric. For this goal we will use the second equation of (15). Thus performing similar calculations we get in the weak sense,

$$(v_0 \cdot \vec{n}) d\sigma_{\partial D} = (b_0 \cdot \vec{n}) d\sigma_{\partial\Omega} - \big\{2\partial_1 b_0 \cdot \nabla v_0^2 - 2\partial_2 b_0 \cdot \nabla v_0^1\big\}.$$

As the functions $\nabla v_0, \nabla b_0$ belong to L^p for any finite $p \in (1,\infty)$, the last term appearing between the braces is a function and consequently the preceding equation is equivalent to the conditions

$$v_0 \cdot \vec{n} = 0 \quad \text{on} \quad \partial D; \quad b_0 \cdot \vec{n} = 0 \quad \text{on} \quad \partial\Omega \tag{24}$$

and

$$\partial_1 b_0 \cdot \nabla v_0^2 - \partial_2 b_0 \cdot \nabla v_0^1 = 0, \quad \text{a.e.} \tag{25}$$

Consequently we deduce by Proposition 3.4 that the curve ∂D is a streamline for φ_0 and $\partial\Omega$ is a streamline for ψ_0. Without loss of generality one can assume that the ball Ω is centered at the origin and with radius r. It is known that in this case the stream function φ_0 has the form

$$\varphi_0(x) = \begin{cases} \frac{1}{4}|x|^2 - \frac{r^2}{2}\big(\log\frac{1}{r} + \frac{1}{2}\big), & |x| \le r, \\ \frac{r^2}{2}\ln|x|, & |x| \ge r. \end{cases}$$

From which we deduce that the streamlines of φ_0 are concentric circles and this implies in turn that ∂D has the same center as $\partial\Omega$. We can also deduce that

$$v_0(x) = f(|x|)x^\perp \quad \text{and} \quad b_0(x) = g(|x|)x^\perp$$

for two known Lipschitz functions f and g. To check the equation (25) we shall write it in the weak sense and use the foregoing structure for v_0 and b_0,

$$\begin{aligned}
\partial_1 b_0 \cdot \nabla v_0^2 - \partial_2 b_0 \cdot \nabla v_0^1 &= \operatorname{div}\big(v_0^2 \partial_1 b_0 - v_0^1 \partial_2 b_0\big) \\
&= \operatorname{div}\big(r f(r) \partial_r (g(r) x^\perp)\big) \\
&= \operatorname{div}\big(f(r)(r\partial_r g + g) x^\perp\big) \\
&= 0.
\end{aligned}$$

This concludes the proof in the case of disjoint C^1 boundaries.

• **Rough curves.** We start with writing in the distribution sense the equations of the stationary patches according to the equations (15). The first one reads as follows

$$\operatorname{div}(v\,\chi_\Omega) = \operatorname{div}(b\,\chi_D).$$

We can readily check in view of the incompressibility condition that the supports of these distributions satisfy

$$\operatorname{supp}(\operatorname{div}(v\,\chi_\Omega)) \subset \partial\Omega \quad \text{and} \quad \operatorname{supp}(\operatorname{div}(v\,\chi_D)) \subset \partial D.$$

Since $\partial D \cap \partial\Omega = \varnothing$ we get

$$\operatorname{div}(v\,\chi_\Omega) = \operatorname{div}(b\,\chi_D) = 0.$$

This means that for any $\psi \in \mathcal{D}(\mathbb{R}^2)$, we have

$$\int_\Omega \operatorname{div}(v\,\psi)dx = \int_D \operatorname{div}(b\psi)dx = 0.$$

The next goal is to deduce from these equations that the Newtonian potential φ_0 and ψ_0 introduced in the previous case are constant on the corresponding boundaries. With this in hand we can conclude by using Fraenkel's result and deduce that the boundaries are necessary circles. Afterwards we shall check by using similar arguments as previously that their centers must agree. As we have mentioned before the major difficulty concerns the use of Gauss-Green formula for rough boundaries. Our approach is based on the use the conformal mapping combined with an approximation procedure. Let $\mathbb{D}$ denote the unit open ball and $\Phi : \mathbb{D} \to \Omega$ be a conformal mapping. Since $\partial\Omega$ is a Jordan curve then by Carathéodory theorem Φ has a continuous extension to $\overline{\mathbb{D}}$ and maps the unit circle $\mathbb{T}$ one-to-one onto $\partial\Omega$. Moreover, according to Theorem 5.4 as the boundary is rectifiable Jordan curve the derivative Φ' exists for almost all $\xi \in \mathbb{T}$ and

$$\Phi' \in L^1(\mathbb{T}).$$

First recall the Cauchy-Riemann equations for the holomorphic function $\Phi = \Phi_1 + i\Phi_2$ inside $\mathbb{D}$

$$\partial_1\Phi_1 = \partial_2\Phi_2, \quad \partial_2\Phi_1 = -\partial_1\Phi_2.$$

Set $F \triangleq v\,\psi$, then we have the equation

$$\int_\Omega \operatorname{div} F\; dx = 0. \tag{26}$$

Observe that from Biot-Savart law the velocity enjoys the following regularities

$$v \in L^\infty,\ \nabla v \in L^p,\ \forall p \in]1, \infty[.$$

For each $0 < r < 1$, let us denote by $\mathbb{D}_r$ the open ball of radius r and centered at the origin and set

$$\Omega_r \triangleq \Phi(\mathbb{D}_r).$$

The boundary $\partial\Omega_r$ is an analytic curve and Φ maps conformally $\mathbb{D}_r$ onto Ω_r. By the change of variable $x = \Phi(y)$ we obtain

$$\int_{\Omega_r} \operatorname{div} F \; dx = \int_{\mathbb{D}_r} (\operatorname{div} F)(\Phi(y)) \; |J_\Phi(y)| dy. \tag{27}$$

Using Cauchy-Riemann equations we obtain the following formula for the Jacobian $|J_\Phi(y)|$,

$$|J_\Phi(y)| = (\partial_1\Phi_1(y))^2 + (\partial_2\Phi_1(y))^2, \quad \forall\, y \in \mathbb{D}.$$

Now we claim that,

$$\begin{aligned}(\operatorname{div} F)(\Phi(y)) \; |J_\Phi(y)| &= \partial_1\Phi_1(\operatorname{div}(F(\Phi(y)) + \partial_1\Phi_2 \operatorname{curl}(F(\Phi(y)) \\ &= \partial_2\Phi_2\partial_1(F_1(\Phi(y)) - \partial_1\Phi_2\partial_2(F_1(\Phi(y)) \\ &+ \partial_1\Phi_1\partial_2(F_2(\Phi(y)) - \partial_2\Phi_1\partial_1(F_2(\Phi(y)).\end{aligned} \tag{28}$$

Indeed, easy computations yield

$$\begin{cases} \partial_1(F_j(\Phi)) = (\partial_1 F_j)(\Phi)\partial_1\Phi_1 + (\partial_2 F_j)(\Phi)\partial_1\Phi_2 \\ \partial_2(F_j(\Phi)) = (\partial_1 F_j)(\Phi)\partial_2\Phi_1 + (\partial_2 F_j)(\Phi)\partial_2\Phi_2. \end{cases}$$

Using Cauchy-Riemann equations one finds

$$\begin{cases} (\partial_1 F_j)(\Phi)|J_\Phi| = \partial_1\Phi_1\partial_1(F_j(\Phi)) - \partial_1\Phi_2\partial_2(F_j(\Phi)) \\ (\partial_2 F_j)(\Phi)|J_\Phi| = \partial_2\Phi_2\partial_2(F_j(\Phi)) - \partial_2\Phi_1\partial_1(F_j(\Phi)). \end{cases}$$

This gives the identity (28). Combining (27) and (28) with Gauss-Green formula we get

$$\begin{aligned}\int_{\Omega_r} \operatorname{div} F \; dx &= r\int_0^{2p} \Big((\partial_2\Phi_2)(re^{i\theta})\cos\theta - (\partial_1\Phi_2)(re^{i\theta})\sin\theta\Big) F_1(\Phi(re^{i\theta}))d\theta \\ &+ r\int_0^{2p} \Big((\partial_1\Phi_1)(re^{i\theta})\sin\theta - (\partial_2\Phi_1)(re^{i\theta})\cos\theta\Big) F_2(\Phi(re^{i\theta}))d\theta.\end{aligned}$$

Recall that

$$F = v\psi = \psi\,\nabla^\perp\varphi_0$$

and thus with the notation $\zeta = e^{i\theta}$ we get

$$\begin{aligned}\int_{\Omega_r} \operatorname{div} F \; dx &= -r\int_0^{2p} \Big((\partial_2\Phi_2)(r\zeta)\cos\theta - (\partial_1\Phi_2)(r\zeta)\sin\theta\Big)(\partial_2\varphi_0)(\Phi(r\zeta))\psi(\Phi(r\zeta))d\theta \\ &+ r\int_0^{2p} \Big((\partial_1\Phi_1)(r\zeta)\sin\theta - (\partial_2\Phi_1)(r\zeta)\cos\theta\Big)(\partial_1\varphi_0)(\Phi(r\zeta))\psi(\Phi(r\zeta))d\theta \\ &= -r\int_0^{2p} \Big((\partial_2\Phi_1)(r\zeta)(\partial_1\varphi_0)(\Phi(r\zeta)) + (\partial_2\Phi_2)(r\zeta)(\partial_2\varphi_0)(\Phi(r\zeta))\Big)\cos\theta\,\psi(\Phi(r\zeta))d\theta \\ &+ r\int_0^{2p} \Big((\partial_1\Phi_1)(r\zeta)(\partial_1\varphi_0)(\Phi(r\zeta)) + (\partial_1\Phi_2)(r\zeta)(\partial_2\varphi_0)(\Phi(r\zeta))\Big)\sin\theta\,\psi(\Phi(r\zeta))d\theta \\ &= -\int_0^{2p} \frac{d}{d\theta}\big\{\varphi_0(\Phi(re^{i\theta}))\big\}\psi(\Phi(re^{i\theta}))d\theta.\end{aligned}$$

We have used the notation,

$$\varphi_0(\Phi(re^{i\theta})) = \varphi_0\big(Q_1(r\cos\theta, r\sin\theta), Q_2(r\cos\theta, r\sin\theta)\big).$$

To pass to the limit in the left-hand side when r approaches 1 we use that

$$\text{div F} = v\cdot\nabla\psi \in L^\infty$$

combined with the fact that the area of Ω_r converges to the area of Ω. This latter claim follows from the formula

$$|\Omega\backslash\Omega_r| = \int_{\mathbb{D}\backslash\mathbb{D}_r} |\Phi'(z)| dA(z).$$

Consequently

$$\begin{aligned} \lim_{r\to 1}\int_{\Omega_r} \operatorname{div} F\, dx &= \int_{\Omega} \operatorname{div} F\, dx \\ &= 0. \end{aligned} \tag{29}$$

Concerning the passage to the limit in the right-hand side we use $\nabla\varphi_0 \in \mathcal{C}_b(\mathbb{R}^2)$ combined with the following result discussed before in the preliminaries,

$$\lim_{r\to 1}\int_0^{2p} |\nabla\Phi(re^{i\theta}) - \nabla\Phi(e^{i\theta})| d\theta = 0.$$

Therefore we obtain from the preceding identity combined with (29)

$$\int_0^{2p} \frac{d}{d\theta}\big\{\varphi_0(\Phi(e^{i\theta}))\big\}\psi(\Phi(e^{i\theta}))d\theta = 0, \quad \forall\, \psi \in \mathcal{D}(\mathbb{R}^2). \tag{30}$$

Let $h : \mathbb{T} \to \mathbb{C}$ be any continuous function on the circle. From Carathéodory Theorem we can extend Φ to the closure $\overline{\mathbb{D}}$ and $\Phi : \overline{\mathbb{D}} \to \overline{\Omega}$ is a homeomorphism. Consequently the inverse $\Phi^{-1} : \partial\Omega \to \mathbb{T}$ is continuous and we can define the function $\psi : \partial\Omega \to \mathbb{C}$ by

$$\psi(z) = h(\Phi^{-1}(z), \quad z \in \partial\Omega.$$

The function ψ is continuous over $\partial\Omega$ and has an extension belonging to $\mathcal{C}_c^\infty(\mathbb{R}^2)$. Therefore we deduce from (30) that for any $h \in \mathcal{C}(\mathbb{T};\mathbb{C})$,

$$\int_0^{2p} \frac{d}{d\theta}\big\{\varphi_0(\Phi(e^{i\theta}))\big\} h(e^{i\theta}) d\theta = 0.$$

This allows to conclude that

$$\frac{d}{d\theta}\big\{\varphi_0(\Phi(e^{i\theta}))\big\} = 0, \quad \text{a.e.}$$

As $\theta \mapsto \frac{d}{d\theta}\big\{\varphi_0(\Phi(e^{i\theta}))\big\}$ is absolutely continuous then we can use Taylor formula which implies the existence of a constant λ such that

$$\varphi_0(\Phi(e^{i\theta})) = \lambda, \quad \forall \theta \in [0, 2p].$$

This means that

$$\varphi_0(z) = \lambda, \quad \forall z \in \partial\Omega.$$

At this stage we can use Fraenkel's result to conclude that the domain Ω should be a ball. The same proof shows that D is also a ball and to check that the balls have the same center we follow the same computations as for the smooth boundaries.

□

6. Generalized vortex patches

In this section, we shall extend the conclusion of Theorem 1.4 to more general initial data belonging to the Yudovich class.

6.1. **General statement.** Before stating our result we shall recall some definitions that were briefly introduced in Section 2. For a continuous vector field $X : \mathbb{R}^2 \to \mathbb{R}^2$ and $\delta \geq 0$ we denote by

$$\mathcal{Z}_X^\delta \triangleq \big\{x \in \mathbb{R}^2, \quad |X(x)| \leq \delta\big\}.$$

Let $\varepsilon \in]0,1[$ and $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two functions. We define the C^ε singular support of the couple (f, g) by

$$\Sigma_{\text{sing}}^\varepsilon(f,g) \triangleq \Sigma_{\text{sing}}^\varepsilon(f) \cup \Sigma_{\text{sing}}^\varepsilon(g),$$

where the singular support of a single function was given in Definition 2.2. We may also recall the push-forward of a vector field X_0 by the flow map associated to another time dependent vector field v, as the solution of the transport equation

$$D_t X = X \cdot \nabla v, \quad X(0) = X_0.$$

Finally recall that the anisotropic space W_X^p was introduced in Definition 2.1. Our main result reads as follows.

Theorem 6.1. *Let $p \in]2, \infty[$, $\varphi_0 : \mathbb{R}^2 \to \mathbb{R}$ be an element of $W^{2,\infty}$ and $G : \mathbb{R} \to \mathbb{R}$ be a function such that $G' \in W^{2,\infty}$ and $\inf_{\mathbb{R}} |G'| > 0$. Take $X_0 = \nabla^\perp \varphi_0$ and $b_0 = \nabla^\perp\{G(\varphi_0)\}$, and assume that the initial data v_0 and b_0 satisfy*

1. *The vorticity ω_0 and the current density j_0 belong to $W_{X_0}^p$.*
2. *There exists a smooth compactly supported function ρ such that $(1-\rho)\omega_0, (1-\rho)j_0 \in C^{1-\frac{2}{p}}$.*
3. *The singular set $\Sigma_{\text{sing}}^{1-\frac{2}{p}}(\omega_0, j_0)$ is compact and there exists $\delta > 0$ such that*

$$\operatorname{dist}\Big(\varphi_0(\mathcal{Z}_{X_0}^\delta), \varphi_0\big(\Sigma_{\text{sing}}^{1-\frac{2}{p}}(\omega_0, j_0)\big)\Big) > 0. \tag{31}$$

Then there exists $T > 0$ and a unique solution (v, b) for the MHD equations with

$$\forall t \in [0,T], \quad \omega(t), j(t) \in W_{X(t)}^p \quad \text{and} \quad X, v, b \in L^\infty([0,T]; \text{Lip}).$$

Moreover, let ψ be the flow of v, then

$$\partial_{X_0}\psi(t) \in L^\infty([0,T]; \text{Lip}).$$

Remark 6.2. *Let $\omega_0 = \chi_\Omega$, $j_0 = \chi_D$ and b_0 the magnetic field associated to j_0. Then following the proof of Theorem 5.1 we obtain that $\omega_0, j_0 \in W_{b_0}^p$ if and only if Ω and D are concentric discs.*

Remark 6.3. *Theorem 6.1 allows to work with more general vortices than the vortex patches: we can for example take an initial vorticity of the form*

$$\omega_0(x) = f(x)\chi_\Omega$$

with f a smooth compactly supported function and the magnetic field b_0 should of course satisfy the assumption (31).

As we shall see now this theorem allows to get the result of Theorem 1.4.

6.2. **Proof of Theorem 1.4.** We shall apply the preceding theorem with $\omega_0 = \chi_\Omega$ and $G(x) = x$, meaning in this case that $X_0 = b_0$. According to the assumption $j_0 \in L^1 \cap W^{1,p}$ and the embedding $W^{1,p} \hookrightarrow L^\infty$ we easily deduce that $j_0 \in W_{X_0}^p$. The assumption $\omega_0 \in W_{X_0}^p$ is equivalent to the vanishing of the normal component of the magnetic field: $b_0 \cdot n = 0$ on the boundary $\partial\Omega$. Therefore it remains to check that the condition (8) of Theorem 1.4 implies the assumption (31).

From the embedding $W^{1,p} \hookrightarrow C^{1-\frac{2}{p}}$, the $C^{1-\frac{2}{p}}$-singular support of j_0 is empty and thus the joint singular support $\Sigma_{\text{sing}}^\varepsilon(\omega_0, j_0)$ coincides with $\partial\Omega$.

Let $x \in \mathcal{Z}_{X_0}^\delta$ and $y \in \partial\Omega$. Since $|b_0(x)| \leq \delta$ then from the condition (8) of Theorem 1.4 we obtain

$$|\varphi_0(x) - \lambda| \geq \eta,$$

with λ the constant value of φ on the boundary $\partial\Omega$. It follows that

$$\forall x \in \mathcal{Z}^{\delta}_{X_0}, \forall y \in \partial\Omega, \quad |\varphi_0(x) - \varphi_0(y)| \geq \eta.$$

This gives the assumption (31) and thus we can apply Theorem 6.1 leading to the first part of Theorem 1.4. Concerning the persistence regularity for the boundary $\psi(t, \partial\Omega)$, recall that a parametrization of Ω is given by the equation

$$\partial_s\gamma_0(s) = X_0(\gamma_0(s)), \quad \gamma_0(0) = x_0 \in \partial\Omega.$$

Therefore we may parametrize $\psi(t, \partial\Omega)$ by $\gamma_t : s \mapsto \psi(t, \gamma_0(s))$ and thus

$$\partial_s\gamma_t(s) = (\partial_{X_0}\psi)(t, \gamma_0(s)).$$

This implies $\partial_s\gamma_t \in W^{1,\infty}$ and consequently $\gamma_t \in W^{2,\infty}$.

6.3. **Persistence of the co-normal regularity.** Next, we shall study the persistence regularity of the solutions in the anisotropic spaces W^p_X and C^ε_X. This step requires that the first commutator between the vector field X_0 and the magnetic field b_0 vanishes an we believe that this algebraic condition is not just a technical artifact but a deep geometric obstruction for the well-posedness problem. We intend to prove the following results.

Proposition 6.4. *Let $\varphi_0 : \mathbb{R}^2 \to \mathbb{R}$ be an element of $W^{2,\infty}$ and $G : \mathbb{R} \to \mathbb{R}$ be a function such that $G' \in W^{2,\infty}$ with $\inf_{\mathbb{R}} |G'| > 0$. Take $X_0 = \nabla^\perp\varphi_0$ and $b_0 = \nabla^\perp\{G(\varphi_0)\}$, and assume that the initial data $\omega_0, j_0 \in W^p_{X_0}$, with $p \in]2, \infty[$. Let (v, b) be a smooth solution of the system (2) defined in some interval $[0, T]$, with $T \leq 1$ and (X_t) be the push-forward of the vector field X_0. Then for any $t \in [0, T]$,*

$$\|(\partial_X\omega, \partial_X j)(t)\|_{L^p} \leq C_0\big(1 + tW^2(t)\big)e^{CtW(t)}$$

and

$$\|(\partial_X v, \partial_X b)(t)\|_{C^{1-\frac{2}{p}}} + \|X_t\|_{C^{1-\frac{2}{p}}} \leq C_0\big(1 + tW^2(t)\big)e^{\exp C_0 tW(t)}.$$

Moreover,

$$\|\partial_X v\|_{W^{1,p}} + \|\partial_X b\|_{W^{1,p}} \leq C_0\big(1 + W(t)\big)e^{CtW(t)}.$$

with

$$W(t) \triangleq \|\nabla v\|_{L^\infty_t L^\infty} + \|\nabla b\|_{L^\infty_t L^\infty}.$$

Proof. Denote by $\mathcal{L} \triangleq \nabla^\perp\Delta^{-1}$, then from Biot-Savart law we can easily check that

$$\partial_X v = \mathcal{L}(\partial_X\omega) - [\mathcal{L}, \partial_X]\omega.$$

To estimate the first term of the right-hand side we combine the dyadic partition of the unity with Bernstein inequality leading for $p \in [1, \infty]$ to

$$\begin{aligned} \|\mathcal{L}\partial_X\omega\|_{C^{1-\frac{2}{p}}} &\lesssim \|\Delta_{-1}\nabla^\perp\Delta^{-1}\partial_X\omega\|_{L^\infty} + \|\partial_X\omega\|_{C^{-\frac{2}{p}}} \\ &\lesssim \|\nabla^\perp\Delta^{-1}\partial_X\omega\|_{L^p} + \|\partial_X\omega\|_{C^{-\frac{2}{p}}}. \end{aligned}$$

According to Lemma 3.5 the vector field X_t remains solenoidal and as Riesz transforms are continuous over L^p for $p \in]1, \infty[$, then

$$\begin{aligned} \|\nabla^\perp\Delta^{-1}\partial_X\omega\|_{L^p} &= \|\nabla^\perp\Delta^{-1}\mathrm{div}\,(X\omega)\|_{L^p} \\ &\lesssim \|X\omega\|_{L^p} \\ &\lesssim \|X\|_{L^\infty}\|\omega\|_{L^p}. \end{aligned}$$

By the virtue of Lemma 7.3, one obtains

$$\big\|[\mathcal{L}, \partial_X]\omega\big\|_{C^{1-\frac{2}{p}}} \lesssim \|X\|_{C^{1-\frac{2}{p}}}\|\omega\|_{L^\infty\cap L^p}.$$

Putting together the preceding estimates yields

$$\|\partial_X v\|_{C^{1-\frac{2}{p}}} \lesssim \|\partial_X \omega\|_{C^{-\frac{2}{p}}} + \|X\|_{C^{1-\frac{2}{p}}} \|\omega\|_{L^\infty\cap L^p}. \tag{32}$$

Next we shall estimate $\|X(t)\|_{C^{1-\frac{2}{p}}}$. For this purpose we use the persistence result of Proposition 2.5 which gives for $p \in (2,\infty)$

$$\|X(t)\|_{C^{1-\frac{2}{p}}} \le Ce^{CV(t)}\Big(\|X_0\|_{C^{1-\frac{2}{p}}} + \int_0^t e^{-CV(\tau)}\|\partial_X v(\tau)\|_{C^{1-\frac{2}{p}}} d\tau\Big), \tag{33}$$

with $V(t) = \|\nabla v\|_{L^1_t L^\infty}$. Set $f(t) = e^{-CV(t)}\|\partial_X v\|_{C^{1-\frac{2}{p}}}$, then

$$f(t) \lesssim \|\partial_X \omega(t)\|_{C^{-\frac{2}{p}}} + \|X_0\|_{C^{1-\frac{2}{p}}}\|\omega(t)\|_{L^p\cap L^\infty} + \|\omega(t)\|_{L^p\cap L^\infty}\int_0^t f(\tau)d\tau.$$

This gives in view of Gronwall inequality,

$$f(t) \lesssim \Big(\|\partial_X \omega\|_{L^\infty_t C^{-\frac{2}{p}}} + \|X_0\|_{C^{1-\frac{2}{p}}}\|\omega\|_{L^\infty_t(L^p\cap L^\infty)}\Big)e^{Ct\|\omega\|_{L^\infty_t(L^p\cap L^\infty)}}.$$

Set $W(t) \triangleq \|\nabla v\|_{L^\infty_t L^\infty} + \|\nabla b\|_{L^\infty_t L^\infty}$ then we obtain from Proposition 4.2

$$\|\omega\|_{L^\infty_t(L^p\cap L^\infty)} \le C_0 e^{CtW(t)} + C_0 t W^2(t).$$

Therefore we get by restricting $t \in [0,1]$

$$\|\partial_X v(t)\|_{C^{1-\frac{2}{p}}} \lesssim \Big(\|\partial_X \omega(t)\|_{C^{-\frac{2}{p}}} + C_0 + C_0 t W^2(t)\Big)e^{\exp C_0 t W(t)}.$$

Performing the same analysis for the magnetic field we get

$$\|\partial_X b(t)\|_{C^{1-\frac{2}{p}}} \lesssim \Big(\|\partial_X j(t)\|_{C^{-\frac{2}{p}}} + C_0 + C_0 t W^2(t)\Big)e^{\exp C_0 t W(t)}$$

and consequently

$$\|(\partial_X v, \partial_X b)(t)\|_{C^{1-\frac{2}{p}}} \lesssim \Big(\|(\partial_X \omega, \partial_X j)(t)\|_{C^{-\frac{2}{p}}} + C_0 + C_0 t W^2(t)\Big)e^{\exp C_0 t W(t)}. \tag{34}$$

To estimate the co-normal regularity of ω and j we shall first write down the equations of $\partial_X \omega$ and $\partial_X j$. According to Proposition 3.2 and using the equations (15) we get

$$\begin{cases} D_t \partial_X \omega = \partial_X(b\cdot\nabla j) \\ D_t \partial_X j = \partial_X(b\cdot\nabla\omega) + 2\partial_X(\partial_1 b\cdot\nabla v^2 - \partial_2 b\cdot\nabla v^1). \end{cases} \tag{35}$$

From the relation (12), we get

$$\partial_X\partial_b - \partial_b\partial_X = \partial_{\partial_X b - \partial_b X}.$$

Consequently,

$$\begin{cases} D_t \partial_X \omega = b\cdot\nabla\partial_X j + \partial_{\partial_X b - \partial_b X}\, j \\ D_t \partial_X j = b\cdot\nabla\partial_X\omega + \partial_{\partial_X b - \partial_b X}\,\omega + 2\partial_X(\partial_1 b\cdot\nabla v^2 - \partial_2 b\cdot\nabla v^1) \end{cases} \tag{36}$$

Set $Y_t = \partial_{X_t} b_t - \partial_{b_t} X_t$ then we can easily check that from our choice we obtain at time zero

$$Y_0 = 0.$$

Now according to Lemma 3.1, we get

$$Y_t = 0, \quad \forall t \in [0,T].$$

Therefore equations (36) become

$$\begin{cases} D_t \partial_X \omega = b\cdot\nabla\partial_X j \\ D_t \partial_X j = b\cdot\nabla\partial_X\omega + 2\partial_X(\partial_1 b\cdot\nabla v^2 - \partial_2 b\cdot\nabla v^1). \end{cases}$$

The estimate of the last term of the second equation in the Hölder space with negative index $C^{-\frac{2}{p}}$ is quite difficult due to the fact that the product $L^\infty \times C^{-\frac{2}{p}}$ is not contained in $C^{-\frac{2}{p}}$. To avoid this technical difficulty we shall replace the space $C^{-\frac{2}{p}}$ by Lebesgue space L^p which scales at the same level and satisfies $L^p \hookrightarrow C^{1-\frac{2}{p}}$. Using Proposition 2.6 one gets

$$\|(\partial_X\omega, \partial_X j)(t)\|_{L^p} \le C_0 + C\int_0^t \Big(\|\partial_X \nabla b(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty} + \|\partial_X \nabla v(\tau)\|_{L^p}\|\nabla b(\tau)\|_{L^\infty}\Big)d\tau. \tag{37}$$

From Biot-Savart law we have easily

$$\|\partial_X \nabla b\|_{L^p} \le \sum_{i,k=1}^{2} \|\partial_X \mathcal{R}_{i,k} j\|_{L^p} \quad \text{and} \quad \|\partial_X \nabla v\|_{L^p} \le \sum_{i,k=1}^{2} \|\partial_X \mathcal{R}_{i,k}\omega\|_{L^p}$$

where $\mathcal{R}_{i,k} = \partial_i\partial_j\Delta^{-1}$. Now we shall combine the identity

$$\partial_X \mathcal{R}_{i,k}\omega = \mathcal{R}_{i,k}\big(\partial_X\omega\big) - \big[\mathcal{R}_{i,k}, \partial_X\big]\omega$$

together with the continuity of Riesz transforms on the L^p spaces with $p \in (1,\infty)$ leading finally to

$$\|\partial_X \mathcal{R}_{i,k}\omega\|_{L^p} \lesssim \|\partial_X\omega\|_{L^p} + \big\|[\mathcal{R}_{i,k}, \partial_X]\omega\|_{L^p}.$$

At this stage we shall use Calderón's estimate, see Lemma 7.2,

$$\big\|[\mathcal{R}_{i,j}, \partial_X]\omega\|_{L^p} \le C\|\nabla X\|_{L^\infty}\|\omega\|_{L^p}. \tag{38}$$

It is not at all obvious how to bound the Lipschitz norm of the vector field X from its evolution equation due to the low regularity of v and as we shall see its specific structure will be of great importance to reach this target. Indeed, we know that at time zero the magnetic field is given by $b_0 = \nabla^\perp\{G(\varphi_0)\}$. Thus it follows from Lemma 3.5 that

$$b_t = \nabla^\perp\{G(\varphi(t))\} \quad \text{and} \quad X_t = \nabla^\perp\varphi_t \tag{39}$$

with φ_t the unique solution of the transport equation

$$D_t\varphi = 0, \quad \varphi(0) = \varphi_0.$$

Therefore we get the relation $b_t = G'(\varphi_t)X_t$ and thus differentiating with respect to the spatial variable we obtain

$$\|\nabla X_t G'(\varphi)\|_{L^\infty} \le \|\nabla b_t\|_{L^\infty} + \|X_t\|_{L^\infty}^2\|G''\|_{L^\infty}.$$

By the assumptions G'' is bounded and $|G'|$ is bounded below by a positive constant which imply

$$\|\nabla X_t\|_{L^\infty} \lesssim \|\nabla b_t\|_{L^\infty} + \|X_t\|_{L^\infty}^2.$$

Coming back to the equation (14) and using the maximum principle and Gronwall inequality we get easily

$$\begin{aligned}\|X(t)\|_{L^\infty} &\le \|X_0\|_{L^\infty} + \int_0^t \|X(\tau)\|_{L^\infty}\|\nabla v(\tau)\|_{L^\infty}d\tau\\ &\le C_0 e^{CV(t)}.\end{aligned} \tag{40}$$

Hence

$$\|X_t\|_{\text{Lip}} \lesssim \|\nabla b_t\|_{L^\infty} + C_0 e^{CtW(t)}. \tag{41}$$

Plugging this estimate into (38) and using Proposition 4.2 one obtains

$$\begin{aligned}\big\|[\mathcal{R}_{i,j}, \partial_X]\omega(t)\|_{L^p} &\le C_0\big(W(t) + e^{CtW(t)}\big)e^{CtW(t)}\\ &\le C_0\big(W(t)+1\big)e^{CtW(t)}.\end{aligned}$$

Thus we deduce

$$\|\partial_X \mathcal{R}_{i,k}\omega(t)\|_{L^p} \lesssim C_0\big(W(t)+1\big)e^{CtW(t)} + \|\partial_X\omega(t)\|_{L^p}. \tag{42}$$

Similarly we get for the current density the estimate,

$$\|\partial_X \mathcal{R}_{i,k}\, j(t)\|_{L^p} \lesssim C_0\big(W(t)+1\big)e^{CtW(t)} + \|\partial_X\, j(t)\|_{L^p}.$$

Inserting these estimates into (37) yields in view of Gronwall inequality

$$
\begin{aligned}
\|(\partial_X\omega,\partial_X j)(t)\|_{L^p} &\leq C_0\big(tW^2(t)+tW(t)\big)e^{CtW(t)}+\int_0^t W(\tau)\|(\partial_X\omega,\partial_X j)(\tau)\|_{L^p}\\
&\leq C_0\big(1+tW^2(t)\big)e^{CtW(t)}. \qquad (43)
\end{aligned}
$$

Combining this estimate with (34) gives for $t\in[0,1]$

$$
\|(\partial_X v,\partial_X b)(t)\|_{C^{1-\frac{2}{p}}}\leq C_0\big(1+tW^2(t)\big)e^{\exp C_0 tW(t)}.
$$

Putting this estimate in (33) one gets for $t\in[0,1]$,

$$
\|X(t)\|_{C^{1-\frac{2}{p}}}\leq C_0\big(1+tW^2(t)\big)e^{\exp C_0 tW(t)}. \qquad (44)
$$

Now we shall estimate the co-normal regularity of $\partial_X v$ and $\partial_X b$ in $W^{1,p}$. First it is easily seen that for $p\in(1,\infty)$

$$
\begin{aligned}
\|\partial_X v\|_{L^p} &\leq \|X\|_{L^\infty}\|\nabla v\|_{L^p}\\
&\lesssim \|X\|_{L^\infty}\|\omega\|_{L^p}.
\end{aligned}
$$

Denote by $\mathcal{L}\triangleq\nabla^\perp\Delta^{-1}$, then from Biot-Savart law,

$$
\begin{aligned}
\partial_i\partial_X v &= \partial_i X\cdot\nabla v+\partial_X\partial_i\mathcal{L}\omega\\
&= \partial_i X\cdot\nabla v+\partial_i\mathcal{L}\partial_X\omega-\big[\partial_i\mathcal{L},\partial_X\big]\omega.
\end{aligned}
$$

Since Riesz transform $\partial_i\mathcal{L}$ is continuous over L^p then we deduce

$$
\|\partial_i X\cdot\nabla v+\partial_i\mathcal{L}\partial_X\omega\|_{L^p}\lesssim\|\nabla X\|_{L^\infty}\|\omega\|_{L^p}+\|\partial_X\omega\|_{L^p}.
$$

Using Lemma 7.2, we get

$$
\big\|[\partial_i\mathcal{L},\partial_X]\omega\big\|_{L^p}\lesssim\|\nabla X\|_{L^\infty}\|\omega\|_{L^p}.
$$

Putting together the preceding estimates implies

$$
\|\partial_X v\|_{W^{1,p}}\lesssim\|\partial_X\omega\|_{L^p}+\|X\|_{\mathrm{Lip}}\|\omega\|_{L^p}. \qquad (45)
$$

Performing the same computations for the magnetic field gives the estimate

$$
\|\partial_X b\|_{W^{1,p}}\lesssim\|\partial_X j\|_{L^p}+\|X\|_{\mathrm{Lip}}\|j\|_{L^p}. \qquad (46)
$$

Combining Proposition 4.2 with the estimates (45), (46), (41) and (43) gives

$$
\begin{aligned}
\|\partial_X v(t)\|_{W^{1,p}}+\|\partial_X b(t)\|_{W^{1,p}} &\leq C_0\Big(1+tW^2(t)+\|\nabla b(t)\|_{L^\infty}\Big)e^{CtW(t)}\\
&\leq C_0\big(1+W(t)\big)e^{CtW(t)}. \qquad (47)
\end{aligned}
$$

□

6.4. **Persistence of the regularity far from the boundary.** We have seen in the previous section how to propagate the co-normal regularity of the solution using in a crucial way the special structure of the magnetic field which should be tangential to the boundary. However the vector field X_0 is singular at some points far from the boundary and thus we cannot recover the regularity everywhere. The idea to follow is simple: to track the regularity far from the singular set we can use somehow the hyperbolic structure of the equations through the classical principle of finite speed propagation of the smooth part. Even though the equations are not local, we shall prove that the singular set does not affect for small time the smooth part of the solution. Before giving more details we need to recall the following notations:

$$
\mathcal{Z}^\delta_{X_0}\triangleq\big\{x\in\mathbb{R}^2,\quad |X_0(x)|\leq\delta\big\},\quad \Sigma^{1-\frac{2}{p}}_{\mathrm{sing}}\triangleq\Sigma^{1-\frac{2}{p}}_{\mathrm{sing}}(\omega_0)\cup\Sigma^{1-\frac{2}{p}}_{\mathrm{sing}}(j_0)
$$

and

$$W(t) \triangleq \sup_{\tau\in[0,t]} \big(\|\nabla v(\tau)\|_{L^\infty} + \|\nabla b(\tau)\|_{L^\infty}\big), \quad V(t) \triangleq \int_0^t \big(\|\nabla v(\tau)\|_{L^\infty} + \|\nabla b(\tau)\|_{L^\infty}\big) d\tau.$$

Proposition 6.5. *Let v_0 and b_0 be two divergence-free vector fields satisfying the assumptions of Theorem 6.1. Let ψ denote the flow associated to the velocity v. Then there exists a function $(t,x) \mapsto \chi(t,x) \in [0,1]$, taking 1 in a neighborhood of $\psi(t,\mathcal{Z}^\delta)$ and vanishing around $\psi(t,\Sigma_{\text{sing}}^{1-\frac{2}{p}})$ such that: for any T satisfying*

$$C_0 T W^4(T) e^{\exp C_0 T W(T)} \le 1$$

we get

$$\forall t\in[0,T], \quad \|\chi(t)\omega(t)\|_{C^{1-\frac{2}{p}}} + \|\chi(t) j(t)\|_{C^{1-\frac{2}{p}}} \le C_0(1+tW^2(t)) e^{\exp C_0 t W(t)}.$$

Proof. From the assumption (31) and the compactness of the singular set $\Sigma_{\text{sing}}^{1-\frac{2}{p}}$ we can easily prove the existence of small $\delta>0$ depending particularly on $\|\nabla\varphi_0\|_{L^\infty}$ such that

$$\operatorname{dist}\big(\varphi_0(\mathcal{Z}^\delta_{X_0}), \varphi_0(\Sigma_{\text{sing}}^{1-\frac{2}{p},\delta})\big) > 0,$$

where we denote by

$$\Sigma_{\text{sing}}^{1-\frac{2}{p},\delta} \triangleq \Big\{x\in\mathbb{R}^2; d(x,\Sigma_{\text{sing}}^{1-\frac{2}{p}}) \le \delta\Big\}.$$

Using Urysohn Theorem we can construct a smooth function $H:\mathbb{R}\to[0,1]$ such that

$$H(\theta) = \begin{cases} 1, & \text{if} \quad \theta\in\varphi_0(\mathcal{Z}^\delta_{X_0}),\\ 0, & \text{if} \quad \theta\in\varphi_0(\Sigma_{\text{sing}}^{1-\frac{2}{p},\delta}).\end{cases}$$

An explicit formula for this function in the Lipschitz class is given by

$$H(\theta) = \frac{\operatorname{dist}(\theta,A)}{\operatorname{dist}(\theta,A)+\operatorname{dist}(\theta,B)}, \quad A \triangleq \varphi_0(\Sigma_{\text{sing}}^{1-\frac{2}{p},\delta}), \quad B\triangleq \varphi_0(\mathcal{Z}^\delta_{X_0}).$$

We point out that by enlarging a little bit A and B and following a smoothing procedure we can construct H in C^∞ class with bounded derivatives. Now introduce the new function

$$\chi_0(x) = H(\varphi_0(x)). \tag{48}$$

Since φ_0 belongs to $W^{2,\infty}$ and H is very smooth we get $\chi_0\in W^{2,\infty}$. It is easy to check that this function satisfies the following properties:

$$\chi_0(x) = \begin{cases} 1, & \text{if} \quad x\in\mathcal{Z}^\delta_{X_0},\\ 0, & \text{if} \quad x\in\Sigma_{\text{sing}}^{1-\frac{2}{p},\delta}.\end{cases}$$

It is clear that

$$\forall x\in\mathbb{R}^2, \quad 1-\chi_0(x)\neq 0 \Longrightarrow |X_0(x)|>\delta. \tag{49}$$

Moreover according once again to the assumptions of Theorem 6.1, the functions $\chi_0\omega_0$ and $\chi_0 j_0$ belongs to the space $C^{1-\frac{2}{p}}$. Let φ be the solution of the transport equation

$$\begin{cases} D_t\varphi = 0,\\ \varphi(0,x) = \varphi_0(x).\end{cases} \tag{50}$$

then by Lemma 3.5

$$X(t,x) = \nabla^\perp\varphi(t,x).$$

Define the cut-off function

$$\chi(t,x) = H(\varphi(t,x)). \tag{51}$$

Then it is easy seen that,

$$\begin{cases} D_t\chi = 0, \\ \chi(0,x) = \chi_0(x). \end{cases} \tag{52}$$

We shall now prove the following assertion,

$$\forall x \in \mathbb{R}^2, \quad 1-\chi(t,x) \neq 0 \Longrightarrow |X(t,x)| \geq \delta e^{-CV(t)}. \tag{53}$$

Indeed, set $Y(t,x) = X(t,\psi(t,x))$ where ψ is the flow associated to the vector field v. Then

$$\begin{aligned} \partial_t Y(t,x) &= (D_t X)(t,\psi(t,x)) \\ &= Y(t,x)\cdot\{(\nabla v)(t,\psi(t,x))\} \end{aligned}$$

and we get by Gronwall inequality

$$|Y(t,x)| \leq |X_0(x)| e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau}.$$

Combining this estimate with the reversibility of the equation gives

$$|X_0(x)| \leq |Y(t,x)| e^{C\int_0^t \|\nabla v(\tau)\|_{L^\infty} d\tau},$$

which means that

$$|X(t,x)| \geq |X_0(\psi^{-1}(t,x))| e^{-CV(t)}.$$

Consequently, by (49) one gets

$$\forall x \in \mathbb{R}^2, \quad 1-\chi_0(\psi^{-1}(t,x)) \neq 0 \Longrightarrow |X(t,x)| \geq \delta\, e^{-CV(t)}.$$

Since $\chi(t,x) = \chi_0(\psi^{-1}(t,x)$ then the proof of (53) is now complete. The next step is to estimate the regularity of the solutions far from the boundary. To do so we start with the following notations

$$f^-(t,x) = f(t,x)\chi(t,x), \quad f^+(t,x) = f(t,x)(1-\chi(t,x)).$$

Combining (15) and (52), we find that ω^- satisfies the equation

$$D_t\omega^- = \chi b\cdot\nabla j = b\cdot\nabla j^- - j\, b\cdot\nabla\chi.$$

According to (39) we get

$$b(t,x) = \nabla^\perp\{G(\varphi(t,x))\}.$$

which yields in view of (51) to

$$b(t,x)\cdot\nabla\chi(t,x) = 0.$$

Therefore the equation of ω^- becomes

$$D_t\omega^- = b\cdot\nabla j^-. \tag{54}$$

By the same way we can establish that

$$D_t j^- = b\cdot\nabla j^- + 2\chi\big(\partial_1 b\cdot\nabla v^2 - \partial_2 b\cdot\nabla v^1\big). \tag{55}$$

Since $b = b^+ + b^-$ then the last term can be decomposed as follows

$$\begin{aligned} \chi\big(\partial_1 b\cdot\nabla v^2 - \partial_2 b\cdot\nabla v^1\big) &= \chi\big(\partial_1 b^+\cdot\nabla v^2 - \partial_2 b^+\cdot\nabla v^1\big) \\ &+ \chi\big(\partial_1 b^-\cdot\nabla v^2 - \partial_2 b^-\cdot\nabla v^1\big) \\ &\triangleq \mathrm{I} + \mathrm{II}. \end{aligned} \tag{56}$$

Straightforward computations give for the first term

$$\begin{aligned} \mathrm{I} &= \chi(1-\chi)\big(\partial_1 b\cdot\nabla v^2 - \partial_2 b\cdot\nabla v^1\big) \\ &- \chi\big(\partial_1\chi\, b\cdot\nabla v^2 - \partial_2\chi\, b\cdot\nabla v^1\big) \\ &\triangleq \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

To estimate the term I_1 we shall use the identity (20),

$$\begin{aligned} \mathrm{I}_1 &= 2\chi\frac{1-\chi}{|X|^2}\Big\{\partial_X b^1\,\partial_X v^2-\partial_X b^2\,\partial_X v^1\Big\}\\ &+ \frac{1-\chi}{|X|^2}\Big\{j^-\,X\cdot\partial_X v-\omega^-\,X\cdot\partial_X b\Big\}. \end{aligned} \tag{57}$$

Using (53) and the algebra structure of C^ε for $\varepsilon>0$, we get

$$\begin{aligned} \|\mathrm{I}_1(t)\|_{C^{1-\frac2p}} &\leq Ce^{CV(t)}\big(1+\|\chi(t)\|^2_{C^{1-\frac2p}}\big)\|X(t)\|_{C^{1-\frac2p}}\|\partial_X v(t)\|_{C^{1-\frac2p}}|\partial_X b(t)\|_{C^{1-\frac2p}}\\ &+ Ce^{CV(t)}\|\chi(t)\|_{C^{1-\frac2p}}\big(1+\|X(t)\|^2_{C^{1-\frac2p}}\big)\Big\{\|j^-(t)\|_{C^{1-\frac2p}}\|\partial_X v(t)\|_{C^{1-\frac2p}}\\ &\qquad +\|\omega^-(t)\|_{C^{1-\frac2p}}\|\partial_X b(t)\|_{C^{1-\frac2p}}\Big\}. \end{aligned} \tag{58}$$

To estimate $\|\chi\|_{C^{1-\frac2p}}$, we apply Proposition 2.5 to the equation (52),

$$\begin{aligned} \|\chi(t)\|_{C^{1-\frac2p}} &\leq C\|\chi_0\|_{C^{1-\frac2p}}e^{CV(t)}\\ &\leq C_0e^{CtW(t)}. \end{aligned} \tag{59}$$

Combining Proposition 6.4 with (58) and (59) we get for $t\in[0,1]$

$$\begin{aligned} \|\mathrm{I}_1(t)\|_{C^{1-\frac2p}} &\leq C_0(1+tW^2(t))e^{\exp C_0tW(t)}\Big(1+\|j^-(t)\|_{C^{1-\frac2p}}+\|\omega^-(t)\|_{C^{1-\frac2p}}\Big)\\ &\leq C_0e^{\exp C_0tW(t)}\big(1+W(t)\big)\Big(1+\|j^-(t)\|_{C^{1-\frac2p}}+\|\omega^-(t)\|_{C^{1-\frac2p}}\Big). \end{aligned} \tag{60}$$

The term I_2 can be estimated as follows,

$$\|\mathrm{I}_2(t)\|_{C^{1-\frac2p}} \leq C\|\nabla\chi(t)\|_{C^{1-\frac2p}}\|b(t)\|_{C^{1-\frac2p}}\|\chi\nabla v(t)\|_{C^{1-\frac2p}}. \tag{61}$$

For the last term of the right-hand side we write,

$$\begin{aligned} \chi\partial_i v &= \chi\partial_i\nabla^\perp\Delta^{-1}\omega\\ &= \partial_i\nabla^\perp\Delta^{-1}(\omega^-)-\Big[\partial_i\nabla^\perp\Delta^{-1},\chi\Big]\omega. \end{aligned}$$

The first term can be treated by using Bernstein inequality leading to

$$\begin{aligned} \|\partial_i\nabla^\perp\Delta^{-1}(\omega^-)\|_{C^{1-\frac2p}} &\lesssim \|\Delta_{-1}\partial_i\nabla^\perp\Delta^{-1}(\omega^-)\|_{L^\infty}+\|\omega^-\|_{C^{1-\frac2p}}\\ &\lesssim \|\omega^-\|_{L^p}+\|\omega^-\|_{C^{1-\frac2p}}\\ &\lesssim \|\omega\|_{L^p}+\|\omega^-\|_{C^{1-\frac2p}}. \end{aligned}$$

Using Proposition 4.2 we find

$$\|\partial_i\nabla^\perp\Delta^{-1}\omega^-(t)\|_{C^{1-\frac2p}}\leq C_0e^{CtW(t)}+\|\omega^-(t)\|_{C^{1-\frac2p}}.$$

As to the commutator term we use Lemma 7.2,

$$\Big\|\big[\partial_i\nabla^\perp\Delta^{-1},\chi\big]\omega\Big\|_{C^{1-\frac2p}}\lesssim\|\chi\|_{\mathrm{Lip}}\|\omega\|_{L^p}.$$

Since χ is transported by the flow then

$$\begin{aligned} \|\chi(t)\|_{\mathrm{Lip}} &\leq C\|\chi_0\|_{\mathrm{Lip}}e^{C\|\nabla v\|_{L^1_tL^\infty}}\\ &\leq C_0e^{CtW(t)}. \end{aligned}$$

Hence we find using once again Proposition 4.2

$$\Big\|\big[\partial_i\nabla^\perp\Delta^{-1},\chi\big]\omega(t)\Big\|_{C^{1-\frac{2}{p}}}\le C_0e^{CtW(t)}.$$

Putting together the preceding estimates gives

$$\|\chi\nabla v(t)\|_{C^{1-\frac{2}{p}}}\le C_0e^{CtW(t)}+\|\omega^-(t)\|_{C^{1-\frac{2}{p}}}. \tag{62}$$

Coming back to (61), then it remains to estimate $\|b(t)\|_{C^{1-\frac{2}{p}}}$ and $\|\nabla\chi(t)\|_{C^{1-\frac{2}{p}}}$. The first term is estimated as follows,

$$\begin{aligned}\|b(t)\|_{C^{1-\frac{2}{p}}} &\lesssim \|b(t)\|_{L^\infty}+W(t)\\ &\lesssim \|b_0\|_{L^\infty}e^{C\|\nabla v\|_{L^1_tL^\infty}}+W(t)\\ &\lesssim C_0e^{CtW(t)}+W(t).\end{aligned} \tag{63}$$

Concerning the second one $\|\nabla\chi(t)\|_{C^{1-\frac{2}{p}}}$ recall that $\nabla^\perp\varphi(t)=X(t)$ and $\chi(t,x)=H(\varphi(t,x)$ which imply

$$\begin{aligned}\|\nabla^\perp\chi(t)\|_{C^{1-\frac{2}{p}}} &= \|H'(\varphi(t))\,X(t)\|_{C^{1-\frac{2}{p}}}\\ &\le \|H'(\varphi(t))\|_{C^{1-\frac{2}{p}}}\|X(t)\|_{C^{1-\frac{2}{p}}}.\end{aligned}$$

Now we use the classical composition law

$$\|H'(\varphi(t))\|_{C^{1-\frac{2}{p}}}\lesssim\|H'\|_{W^{1,\infty}}\big(1+\|\varphi(t)\|_{C^{1-\frac{2}{p}}}\big)$$

which gives according to Proposition 4.2 and Proposition 6.4 that for $t\in[0,1]$,

$$\begin{aligned}\|\nabla^\perp\chi(t)\|_{C^{1-\frac{2}{p}}} &\le C_0\big(1+tW^2(t)\big)e^{\exp C_0tW(t)}\\ &\le C_0(1+W(t))e^{\exp C_0tW(t)}.\end{aligned} \tag{64}$$

Putting together (61),(62), (63) and (64) we obtain

$$\|\mathrm{I}_2(t)\|_{C^{1-\frac{2}{p}}}\le C_0\big[1+W^2(t)\big]e^{\exp C_0tW(t)}\big(1+\|\omega^-(t)\|_{C^{1-\frac{2}{p}}}\big) \tag{65}$$

Combining this estimate with (60) we get

$$\|\mathrm{I}(t)\|_{C^{1-\frac{2}{p}}}\le C_0\big[1+W^2(t)\big]e^{\exp C_0tW(t)}\Big(1+\|\omega^-(t)\|_{C^{1-\frac{2}{p}}}+\|j^-(t)\|_{C^{1-\frac{2}{p}}}\Big). \tag{66}$$

Coming back to the estimate of the second term II of (56). From the algebra structure of $C^{1-\frac{2}{p}}$,

$$\|\mathrm{II}(t)\|_{C^{1-\frac{2}{p}}}\lesssim\|\nabla b^-(t)\|_{C^{1-\frac{2}{p}}}\|\chi\nabla v(t)\|_{C^{1-\frac{2}{p}}}.$$

Therefore we get according to (62)

$$\|\mathrm{II}(t)\|_{C^{1-\frac{2}{p}}}\lesssim\Big(C_0e^{CtW(t)}+\|\omega^-(t)\|_{C^{1-\frac{2}{p}}}\Big)\|\nabla b^-(t)\|_{C^{1-\frac{2}{p}}}.$$

The last term will be estimated as follows,

$$\begin{aligned}\|\partial_ib^-\|_{C^{1-\frac{2}{p}}} &\le \|\chi\partial_ib\|_{C^{1-\frac{2}{p}}}+\|b\partial_i\chi\|_{C^{1-\frac{2}{p}}}\\ &\le \|\chi\partial_ib\|_{C^{1-\frac{2}{p}}}+\|b\|_{C^{1-\frac{2}{p}}}\|\nabla\chi\|_{C^{1-\frac{2}{p}}}\end{aligned}$$

Using (63) and (64) we obtain

$$\|\partial_ib^-(t)\|_{C^{1-\frac{2}{p}}}\le\|\chi(t)\partial_ib(t)\|_{C^{1-\frac{2}{p}}}+C_0\big[1+W^2(t)\big]e^{\exp C_0tW(t)}.$$

Concerning the estimate of the first term of the right-hand side we imitate the same computation of (62)

$$\|\chi\partial_ib\|_{C^{1-\frac{2}{p}}}\le C_0e^{CtW(t)}+\|j^-\|_{C^{1-\frac{2}{p}}}.$$

It follows that

$$\|\partial_i b^-\|_{C^{1-\frac{2}{p}}} \leq \|j^-\|_{C^{1-\frac{2}{p}}} + C_0\big[1+W^2(t)\big]e^{\exp C_0 tW(t)}.$$

Putting together the previous estimates

$$\|\mathrm{II}\|_{C^{1-\frac{2}{p}}} \leq C_0\Big(1+\|\omega^-(t)\|_{C^{1-\frac{2}{p}}}\Big)\Big(1+\|j^-\|_{C^{1-\frac{2}{p}}}\Big)\big[1+W^2(t)\big]e^{\exp C_0 tW(t)}.$$

Inserting this estimate and (66) into (56) gives

$$\|\chi(t)\big(\partial_1 b\cdot\nabla v^2-\partial_2 b\cdot\nabla v^1\big)(t)\|_{C^{1-\frac{2}{p}}} \leq C_0\Big(1+\|\omega^-(t)\|_{C^{1-\frac{2}{p}}}\Big)\Big(\|j^-\|_{C^{1-\frac{2}{p}}}+1\Big)\big[1+W^2(t)\big]e^{\exp C_0 tW(t)}.$$

Set

$$g(t) \triangleq \|\omega^-(t)\|_{C^{1-\frac{2}{p}}} + \|j^-(t)\|_{C^{1-\frac{2}{p}}},$$

then applying Proposition 2.6 to the system (54) and (55) we get for $t\in[0,1]$,

$$g(t) \leq C_0 e^{\exp C_0 tW(t)}\big[1+tW^2(t)\big] + C_0 e^{\exp C_0 tW(t)}\int_0^t g^2(\tau)\big[1+W^2(\tau)\big]d\tau.$$

It follows that for small time $t\in[0,T]$ such that

$$4C_0^2 T e^{\exp C_0 TW(T)}\big[1+W^2(T)\big]^2 \leq 1 \tag{67}$$

we get

$$g(t) \leq 2C_0 e^{\exp C_0 tW(t)}\big[1+tW^2(t)\big].$$

This completes the proof of the proposition. We point out that as a by-product of (62) one obtains

$$\|\chi(t)\nabla v(t)\|_{C^{1-\frac{2}{p}}} + \|\chi(t)\nabla b(t)\|_{C^{1-\frac{2}{p}}} \leq C_0 e^{\exp C_0 tW(t)}\big[1+tW^2(t)\big]. \tag{68}$$

□

6.5. **Proof of Theorem 6.1.** We shall now discuss the proof of Theorem 6.1. We first establish the suitable a priori estimates and second we sketch the principal ingredients for the construction of the solution in our context. We end with the uniqueness part.

Proof. We shall start with the local a priori estimates.
• *Local a priori estimates.*

We assume that the system (2) admits a smooth solution and we wish to find some a priori estimates. The crucial quantities for the persistence of the regularity are the Lipschitz norms of the velocity and the magnetic field. To estimate the Lipschitz norm of the velocity we shall use (68). Then under the assumption (67)

$$\begin{aligned}\|\nabla v(t)\|_{L^\infty} &\leq \|\chi(t)\nabla v(t)\|_{L^\infty} + \|(1-\chi(t))\nabla v(t)\|_{L^\infty}\\ &\leq \|\chi(t)\nabla v(t)\|_{C^{1-\frac{2}{p}}} + \|(1-\chi(t))\nabla v(t)\|_{L^\infty}\\ &\leq C_0 e^{\exp C_0 tW(t)}\big[1+tW^2(t)\big] + \|(1-\chi(t))\nabla v(t)\|_{L^\infty}\end{aligned}$$

To estimate the last term we shall use the identity (18),

$$\begin{aligned}\|(1-\chi(t))\nabla v(t)\|_{L^\infty} &\leq \Big\|\frac{1-\chi(t)}{|X(t)|^2}\Big\|_{L^\infty}\Big(\|X(t)\|_{L^\infty}\|\partial_X v(t)\|_{L^\infty} + \|X(t)\|_{L^\infty}^2\|\omega(t)\|_{L^\infty}\Big)\\ &\leq \Big\|\frac{1-\chi(t)}{|X(t)|^2}\Big\|_{L^\infty}\Big(\|X(t)\|_{L^\infty}\|\partial_X v(t)\|_{C^{1-\frac{2}{p}}} + \|X(t)\|_{L^\infty}^2\|\omega(t)\|_{L^\infty}\Big).\end{aligned}$$

Using Proposition 6.4 combined with (40), (53) and Proposition 4.2

$$\begin{aligned}\|(1-\chi(t))\nabla v(t)\|_{L^\infty} &\le C_0\big[1+tW^2(t)\big]e^{\exp C_0 tW(t)}\Big(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|\nabla b(\tau)\|_{L^\infty}d\tau\Big)\\ &\le C_0\big[1+tW^2(t)\big]^2 e^{\exp C_0 tW(t)}.\end{aligned}$$

Consequently we obtain

$$\|\nabla v(t)\|_{L^\infty} \le C_0\big[1+tW^2(t)\big]^2 e^{\exp C_0 tW(t)}.$$

In a similar way we get for the magnetic field

$$\|\nabla b(t)\|_{L^\infty} \le C_0\big[1+tW^2(t)\big]^2 e^{\exp C_0 tW(t)}.$$

Thus we find under the assumption (67):

$$\forall t\in[0,T],\quad W(t)\le C_0\big[1+tW^2(t)\big]^2 e^{\exp C_0 tW(t)}.$$

The goal is to find a suitable time existence $T=T(C_0)>0$ subject to the above constraints. We shall look for small T such that $W(T)<2eC_0$. This holds true whenever

$$\big(1+4e^2C_0^2\,T\big)^2 e^{\exp 2eC_0^2 T} < 2e.$$

The existence of such T follows from the continuity in time of left-hand side and the fact that the previous inequality is strict for $T=0$. It remains to check the condition (67). This is true if

$$4C_0^2\big[1+4e^2C_0^2\big]^2 T e^{\exp 2eC_0^2 T} \le 1. \tag{69}$$

To guarantee this last condition we take T sufficiently small. Under this assumption we see from the previous computations in the last sections that $\omega(t), j(t)\in W^p_{X(t)},\quad \forall t\in[0,T]$. Moreover the vector fields v, b and X belong to $L^\infty([0,T];W^{1,\infty})$. To achieve the a priori estimates of Theorem 6.1 it remains to check that $\partial_{X_0}\psi(t)\in L^\infty([0,T];W^{1,\infty})$. For this aim we use the identity

$$\partial_{X_0}\psi(t) = X_t\circ\psi(t).$$

It suffices now to use the fact that X_t and $\psi(t)$ belong both to the Lipschitz class $W^{1,\infty}$.

- *Existence and smoothing procedure.*

To justify rigorously the previous a priori estimates and construct a solution as claimed in Theorem 6.1 we start with smoothing out the initial data as follows

$$v_0^n = v_0\star\eta_n,\quad b_0^n=\nabla^\perp\{G(\varphi_n)\},\quad X_0^n = X_0\star\eta_n,\quad \varphi_n=\varphi\star\eta_n$$

where $\eta_n(x)=n^2\eta(nx)$ is a standard mollifier. From the assumptions we can easily check that for any n,

$$v_0^n, b_0^n\in C^{1+\alpha}$$

for any $\alpha\in(0,1)$. Consequently we can apply the classical theory which ensures for each n the existence and the uniqueness of local solution (v^n,b^n) defined on some interval $[0,T_n]$ and with values in $C^{1+\alpha}$. We shall prove that $\inf_n T_n\ge T>0$ where T is defined in (69) but this does not mean that the bounds are uniform in the classical space $C^{1+\alpha}$. The uniformness in tho space is false but it will be proven in the space of the initial data. Indeed, it suffices to show that the smooth family (v_0^n,b_0^n) satisfies the assumptions of Theorem 6.1 with uniform bounds with respect to n. First we intend to check the first assumption , that is,

$$\partial_{X_0^n}\omega_0^n,\quad \partial_{X_0^n} j_0^n\in L^p$$

with uniform bounds. First observe that the vorticity ω_0^n of v_0^n is given by $\omega_0^n=\omega_0\star\rho_n$ and

$$\partial_{X_0^n}\omega_0^n(x) = \partial_{X_0^n - X_0}\omega_0^n(x) + \partial_{X_0}\omega_0^n(x).$$

The first term can be estimated in a classical way as follows

$$\begin{aligned}\|\partial_{X_0^n - X_0}\omega_0^n\|_{L^p} &\leq \|X_0^n - X_0\|_{L^\infty}\|\nabla\omega_0^n\|_{L^p}\\ &\leq \|\nabla X_0\|_{L^\infty}\||\cdot|\eta_n\|_{L^1}\|\omega_0\|_{L^p}\|\nabla\eta_n\|_{L^1}\\ &\lesssim \|\nabla X_0\|_{L^\infty}\|\omega_0\|_{L^p}.\end{aligned}$$

As regards the second term we write

$$\begin{aligned}\partial_{X_0}\omega_0^n(x) &= \eta_n \star (\partial_{X_0}\omega_0)(x) + n^2\int_{\mathbb{R}^2}[X_0(x) - X_0(y)]\cdot\nabla_y\omega_0(y)\eta(n(x-y))dy\\ &\triangleq \mathrm{I}_n + \mathrm{II}_n.\end{aligned}$$

Using the convolution inequalities we obtain

$$\|\mathrm{I}_n\|_{L^p} \lesssim \|\partial_{X_0}\omega_0\|_{L^p}.$$

Integration by parts combined with the incompressibility of X_0 yields

$$\mathrm{II}_n(x) = n^3\int_{\mathbb{R}^2}[X_0(x) - X_0(y)]\cdot(\nabla_y\eta)(n(x-y))\omega_0(y)dy.$$

Thus we get

$$|\mathrm{II}_n(x)| \leq \|\nabla X_0\|_{L^\infty} n^3\int_{\mathbb{R}^2}|x-y||(\nabla_y\eta)(n(x-y))||\omega_0(y)|dy.$$

From the classical convolution laws one gets

$$\|\mathrm{II}_n\|_{L^p} \leq \|\nabla X_0\|_{L^\infty}\||\cdot|\nabla\eta\|_{L^1}\|\omega_0\|_{L^p}.$$

This achieves the proof of the first assumption of Theorem 6.1.
Let us now check the second assumption of this theorem. We shall show that

$$\sup_{n\in\mathbb{N}^\star}\|(1-\rho)\omega_0^n\|_{C^{1-\frac{2}{p}}} + \sup_{n\in\mathbb{N}^\star}\|(1-\rho)j_0^n\|_{C^{1-\frac{2}{p}}} < \infty.$$

We point out that in the application the function $1-\rho$ is closely related to the function χ_0 introduced in (48) and this latter one belongs to $W^{2,\infty}$. Thus the function ρ should belong to $W^{2,\infty}$ and not more. We write

$$\begin{aligned}(1-\rho(x))\omega_0^n(x) &= \eta_n\star[(1-\rho)\omega_0](x) + \int_{\mathbb{R}^2}[\rho(x)-\rho(y)]\omega_0(y)\eta_n(x-y)dy\\ &\triangleq \mathcal{I}_1(x) + \mathcal{I}_2(x)\end{aligned}$$

From the classical convolution inequalities one gets

$$\|\mathcal{I}_1\|_{C^{1-\frac{2}{p}}} \lesssim \|(1-\rho)\omega_0\|_{C^{1-\frac{2}{p}}}.$$

For the second term we claim that

$$\|\mathcal{I}_2\|_{W^{1,\infty}} \lesssim \|\omega_0\|_{L^\infty}.$$

Indeed, the uniform boundedness is easy to get. Concerning the Lipschitz norm we write

$$\nabla\mathcal{I}_2(x) = \nabla\rho(x)\,\eta_n\star\omega_0(x) + n^3\int_{\mathbb{R}^2}[\rho(x)-\rho(y)]\omega_0(y)(\nabla\eta)\big(n(x-y)\big)dy.$$

Consequently we find

$$\begin{aligned}\|\nabla\mathcal{I}_2\|_{L^\infty} &\leq \|\nabla\rho\|_{L^\infty}\|\omega_0\|_{L^\infty}\|\eta_n\|_{L^1} + \|\nabla\rho\|_{L^\infty}\|\omega_0\|_{L^\infty}\||\cdot|\nabla\eta\|_{L^1}\\ &\lesssim \|\omega_0\|_{L^\infty}.\end{aligned}$$

Concerning the uniform estimate of $\|(1-\rho)j_0^n\|_{C^{1-\frac{2}{p}}}$ it suffices to bound $(1-\rho)b_0^n$ in the Hölder space $C^{2-\frac{2}{p}}$. For this purpose we write by the definition

$$(1-\rho)b_0^n = G'(\varphi_n)[(1-\rho)\nabla^\perp\varphi_n].$$

Using the algebra structure of $W^{2-\frac{2}{p}}$ yields

$$\|(1-\rho)b_0^n\|_{C^{2-\frac{2}{p}}} \lesssim \|G'(\varphi_n)\|_{C^{2-\frac{2}{p}}}\big\|[(1-\rho)\nabla^\perp\varphi_n]\big\|_{C^{2-\frac{2}{p}}}.$$

From the classical law products one obtains

$$\|G'(\varphi_n)\|_{C^{2-\frac{2}{p}}} \lesssim \|G'\|_{W^{2,\infty}}\|\varphi_n\|_{C^{2-\frac{2}{p}}}.$$

Combining this inequality with the convolution laws

$$\|\varphi_n\|_{C^{2-\frac{2}{p}}} \lesssim \|\varphi\|_{C^{2-\frac{2}{p}}}$$

allows to get

$$\|G'(\varphi_n)\|_{C^{2-\frac{2}{p}}} \lesssim \|G'\|_{W^{2,\infty}}\|\varphi\|_{C^{2-\frac{2}{p}}}.$$

On the other hand we have

$$\begin{aligned}
\big\|[(1-\rho)\nabla^\perp\varphi_n]\big\|_{C^{2-\frac{2}{p}}} &\lesssim \big\|[(1-\rho)\nabla^\perp\varphi_n]\big\|_{L^\infty} + \big\|\nabla\rho\nabla^\perp\varphi_n\big\|_{W^{1,\infty}} + \big\|(1-\rho)\nabla^\perp\nabla\varphi_n\big\|_{C^{1-\frac{2}{p}}}\\
&\lesssim \|\rho\|_{W^{2,\infty}}\|\varphi_n\|_{W^{2,\infty}} + \big\|(\mathrm{Id}-\Delta_{-1})[(1-\rho)\nabla^\perp\nabla\varphi_n]\big\|_{C^{1-\frac{2}{p}}}\\
&\lesssim \|\rho\|_{W^{2,\infty}}\|\varphi\|_{W^{2,\infty}} + \big\|(\mathrm{Id}-\Delta_{-1})[(1-\rho)\nabla^\perp\nabla\varphi_n]\big\|_{C^{1-\frac{2}{p}}}.
\end{aligned}$$

As to the last term we transform it into

$$\begin{aligned}
(1-\rho)\nabla^\perp\nabla\varphi_n &= \{\nabla[(1-\rho)\nabla^\perp\varphi]\}\star\eta_n + \int_{\mathbb{R}^2}[\rho(y)-\rho(x)]\nabla^\perp\varphi(y)\nabla\eta_n(x-y)dy\\
&\triangleq \mathcal{J}_1+\mathcal{J}_2.
\end{aligned}$$

Using once again the convolution inequalities we find

$$\big\|\mathcal{J}_1\big\|_{C^{1-\frac{2}{p}}} \lesssim \|(1-\rho)\nabla^\perp\varphi\|_{C^{2-\frac{2}{p}}}.$$

For the term $\mathcal{J}_2$ we write

$$\|(\mathrm{Id}-\Delta_{-1})\mathcal{J}_2\|_{C^{1-\frac{2}{p}}} \lesssim \|\nabla\mathcal{J}_2\|_{L^\infty}.$$

It is easy to check that

$$\partial_i\mathcal{J}_2 = (\partial_i\rho\nabla^\perp\varphi)\star\nabla\eta_n - \partial_i\rho(x)\nabla^\perp\varphi\star\nabla\eta_n + \int_{\mathbb{R}^2}[\rho(y)-\rho(x)]\partial_i\nabla^\perp\varphi(y)\nabla\eta_n(x-y)dy.$$

The first two terms of the right-hand side can be estimated as follows

$$\begin{aligned}
\|(\partial_i\rho\nabla^\perp\varphi)\star\nabla\eta_n - \partial_i\rho(x)\nabla^\perp\varphi\star\nabla\eta_n\|_{L^\infty} &\lesssim \|\nabla(\partial_i\rho\nabla^\perp\varphi)\|_{L^\infty} + \|\nabla\rho\|_{L^\infty}\|\nabla\nabla^\perp\varphi\|_{L^\infty}\\
&\lesssim \|\rho\|_{W^{2,\infty}}\|\varphi\|_{W^{2,\infty}}.
\end{aligned}$$

Concerning the last term we write

$$\begin{aligned}
\Big|\int_{\mathbb{R}^2}[\rho(y)-\rho(x)]\partial_i\nabla^\perp\varphi(y)\nabla\eta_n(x-y)dy\Big| &\leq \|\nabla\rho\|_{L^\infty}\|\nabla\nabla^\perp\varphi\|_{L^\infty}\int_{\mathbb{R}^2}|y-x||\nabla\eta_n(x-y)|dy\\
&\lesssim \|\nabla\rho\|_{L^\infty}\|\varphi\|_{W^{2,\infty}}.
\end{aligned}$$

Therefore we obtain

$$\|\mathcal{J}_2\|_{L^\infty} \lesssim \|\rho\|_{W^{2,\infty}}\|\varphi\|_{W^{2,\infty}}.$$

Putting together the preceding estimates allows to get the uniform estimate

$$\sup_{n\in\mathbb{N}^\star}\|(1-\rho)j_0^n\|_{C^{1-\frac{2}{p}}} < \infty.$$

It remains to check the assumption (31) uniformly with respect to n. This condition should be a little bit clarified since the singular support of (ω_0^n, j_0^n) is smoothed out. We replace in (31) the set $\Sigma_{sing}^{1-\frac{2}{p}}(\omega_0^n, j_0^n)$ by $\tilde{\Sigma}_{sing}^{1-\frac{2}{p}}$ defined as follows: we say that $x \notin \tilde{\Sigma}_{sing}^{1-\frac{2}{p}}$ if and only if there exists a smooth compactly supported function χ with $\chi(x_0) = 1$ such that $\chi\omega_0^n$ and χj_0^n belong to $C^{1-\frac{2}{p}}$ uniformly on n. Performing straightforward calculations one can prove that

$$\tilde{\Sigma}_{sing}^{1-\frac{2}{p}} = \Sigma_{sing}^{1-\frac{2}{p}}(\omega_0, j_0).$$

Now since $\{\varphi_n\}$ converges uniformly to φ we can easily see that the assumption (31) is satisfied for sufficiently large values of n. This achieves the fact that the family $\{v_0^n, b_0^n\}$ is smooth and satisfies the assumptions $(1) - (2) - (3)$ of Theorem 6.1 uniformly with respect to n.

- *Uniqueness part.*

Let $\{(v_i, b_i, p_i), i = 1, 2\}$ be two solutions of the system (2) with the same initial data (v_0, b_0) and belonging to the space $L_T^\infty W^{1,\infty}$ such that $(\omega_i, j_i) \in L_T^\infty(L^1 \cap L^\infty)$. We set $v \triangleq v_1 - v_2, b \triangleq b_1 - b_2$ and $p = p_1 - p_2$. It is known that in general the velocity does not belong to L^2 when its vorticity is only bounded and integrable but belongs to L^p, $\forall p > 2$. However by reproducing the arguments developed in [12] we can show the existence of two vector fields σ_1 and σ_2 solutions of the stationary Euler equations and satisfying in addition $\sigma_i \in C_b^\infty$ and $\nabla\sigma_i \in H^s, \forall s \in \mathbb{R}$, such that the solutions v_i and b_i constructed in the previous step belong to $\sigma_1 + L^2$, $\sigma_2 + L^2$, respectively. Therefore and in order to give a simple proof for the uniqueness part we shall assume that $\sigma_i \equiv 0$.
It is easy to check that (v, b) satisfies the following equations

$$\left\{\begin{array}{l}\partial_t v + v_1 \cdot \nabla v + \nabla p = b_1 \cdot \nabla b - v \cdot \nabla v_2 + b \cdot \nabla b_2\\ \partial_t b + v_1 \cdot \nabla b = b_1 \cdot \nabla v - v \cdot \nabla b_2 + b \cdot \nabla v_2.\end{array}\right. \tag{70}$$

Taking the L^2- inner product of the first equation of (70) with v and of the second equation with b we find after using the incompressibility of the involved vector fields,

$$\frac{1}{2}\frac{d}{dt}\big(\|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\big) = \int_{\mathbb{R}^2} \big\{(b_1 \cdot \nabla b) \cdot v + (b_1 \cdot \nabla v) \cdot b\big\} dx + I(t)$$

with

$$I(t) = \int_{\mathbb{R}^2} \big(-v \cdot \nabla v_2 + b \cdot \nabla b_2\big) \cdot v\, dx + \int_{\mathbb{R}^2} \big(-v \cdot \nabla b_2 + b \cdot \nabla v_2\big) \cdot b\, dx.$$

Integration by parts shows that the first term of the right-hand side vanishes. For the term $I(t)$ one obtains by using successively Hölder and Young inequalities

$$|I(t)| \lesssim \big(\|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\big)\big(\|\nabla v_2(t)\|_{L^\infty} + \|\nabla b_2(t)\|_{L^\infty}\big).$$

Consequently

$$\frac{d}{dt}\big(\|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\big) \lesssim \big(\|v(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2\big)\big(\|\nabla v_2(t)\|_{L^\infty} + \|\nabla b_2(t)\|_{L^\infty}\big)$$

and thus the uniqueness follows from Gronwall inequality. $\square$

7. Commutator estimates

We shall in this section discuss some commutator estimates that most of them were of great use in the previous sections. The first one is technical and whose proof can be found for example in [24].

Lemma 7.1. *Let $(a, b) \in [1, \infty]^2$ such that $a \geq b'$ with $\frac{1}{b} + \frac{1}{b'} = 1$. Given f, g and h three functions such that $\nabla f \in L^a$, $g \in L^b$ and $xh \in L^{b'}$. Then,*

$$\|h \star (fg) - f(h \star g)\|_{L^a} \lesssim \|xh\|_{L^{b'}} \|\nabla f\|_{L^a} \|g\|_{L^b}.$$

Next we intend to recall and precise a crowd of estimates for some commutators of Calderón type. First, we denote by $\mathcal{R}_{ij}$ the iterated Riesz transform

$$\mathcal{R}_{ij} \triangleq \partial_i\partial_j\Delta^{-1}.$$

This operator acts continuously over Lebesgue spaces L^p for $1 < p < \infty$ and has an even kernel which is smooth in $\mathbb{R}^2\backslash\{0\}$ and with zero mean value on the the unit circle.

Lemma 7.2. *Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be two smooth functions. Then the following assertions hold true.*

(1) *For $p \in]1, \infty[$ we have*

$$\big\|[\mathcal{R}_{ij}, f]\partial_k g\big\|_{L^p} \lesssim \|\nabla f\big\|_{L^\infty}\|g\|_{L^p}.$$

(2) *For $\varepsilon \in]0, 1[$ and $p \geq \frac{2}{1-\varepsilon}$, we get*

$$\big\|\big[\mathcal{R}_{ij}, f\big]g\big\|_{C^\varepsilon} \lesssim \big(\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}\big)\|g\|_{L^p}.$$

Proof. (1) This result follows from Theorem 1 of [7], but in this theorem the dependence of the constant with respect to the norm of f is not precised. However we can obtain our estimate from Theorem 2 of the same paper [7] and we shall outline in the next lines how to reduce our problem to this case. Let K_i denote the Kernel of Riesz transform $\mathcal{R}_i \triangleq \partial_i\sqrt{-\Delta}$ which is odd, homogeneous of order $-d$ and belongs to $C^\infty(\mathbb{R}^d\backslash\{0\})$. Now it is easy to check that

$$[\mathcal{R}_i, f]\partial_k g(x) = \int_{\mathbb{R}^2} K_i(x-y)\big(f(y) - f(x)\big)\partial_{y_k} g(y)dy.$$

Thus using integration by parts yields

$$\begin{aligned} [\mathcal{R}_i, f]\partial_k g(x) &= -\mathcal{R}_i(g\partial_{x_k} f) + \int_{\mathbb{R}^2}(\partial_{y_k} K_i)(x-y)\big(f(y) - f(x)\big)g(y)dy \\ &\triangleq \mathrm{I} + \mathrm{II}. \end{aligned}$$

To estimate the first term we use the continuity of $\mathcal{R}_i : L^p \to L^p$ for $p \in]1, \infty[$ and therefore

$$\begin{aligned} \|\mathrm{I}\|_{L^p} &\leq C\|g\partial_{x_k} f\|_{L^p} \\ &\leq C\|\nabla f\|_{L^\infty}\|g\|_{L^p}. \end{aligned}$$

As regards the second term we shall use Theorem 2 of [7] which is valid in our context since the map $x \mapsto \partial_{x_k} K_i(x)$ is even, homogeneous of degree $-d-1$, locally integrable in $\mathbb{R}^d\backslash\{0\}$. Consequently

$$\|\mathrm{II}\|_{L^p} \leq C\|\nabla f\|_{L^\infty}\|g\|_{L^p}.$$

Putting together the previous estimates gives the following result

$$\big\|[\mathcal{R}_i, f]\partial_k g\big\|_{L^p} \lesssim \|\nabla f\big\|_{L^\infty}\|g\|_{L^p}. \tag{71}$$

Now let us come back to the iterative Riesz transform $\mathcal{R}_{ij} = \mathcal{R}_i\mathcal{R}_j$ and write

$$\big[\mathcal{R}_{ij}, f\big]\partial_k g = \mathcal{R}_j\big\{\big[\mathcal{R}_i, f\big]\partial_k g\big\} + [\mathcal{R}_j, f]\partial_k\mathcal{R}_i g.$$

Using the preceding result (71) combined with the continuity of Riesz transforms on L^p lead to the desired result.

(2) We shall use the para-differential calculus through Bony's decomposition,

$$\begin{aligned} \big[\mathcal{R}_{ij}, f\big]g &= \sum_{q\in\mathbb{N}}\big[\mathcal{R}_{ij}, S_{q-1}f\big]\Delta_q g + \sum_{q\in\mathbb{N}}\big[\mathcal{R}_{ij}, \Delta_q f\big]S_{q-1}g + \sum_{q\geq -1}\big[\mathcal{R}_{ij}, \Delta_q f\big]\tilde{\Delta}_q g \\ &\triangleq \sum_{q\in\mathbb{N}}\pi_1^q + \sum_{q\in\mathbb{N}}\pi_2^q + \sum_{q\geq -1}\pi_3^q \\ &\triangleq \pi_1 + \pi_2 + \pi_3. \end{aligned}$$

To estimate the first term π_1^q we use its convolution structure,

$$\pi_1^q = h_q \star (S_{q-1}f\Delta_q g) - S_{q-1}f\,(h_q \star \Delta_q g),$$

where $\widehat{h}_q(\xi) = \frac{\xi_i\xi_j}{|\xi|^2}\psi(2^{-q}\xi)$ and ψ is a smooth function supported in an annulus centered at zero. Therefore $h_q(x) = 2^{2q}h(2^q x)$ with $h \in \mathcal{S}$. Then in view of the Lemma 7.1 we get,

$$\|\pi_1^q\|_{L^\infty} \lesssim \|xh_q\|_{L^1}\|\nabla S_{q-1}f\|_{L^\infty}\|\Delta_q g\|_{L^\infty}.$$

Using the fact $\|xh_q\|_{L^1} = 2^{-q}\|xh\|_{L^1}$ combined with Bernstein inequality we obtain with the assumption $p \geq \frac{2}{1-\varepsilon}$

$$\begin{aligned} 2^{q\varepsilon}\|\pi_1^q\|_{L^\infty} &\lesssim 2^{q(-1+\frac{2}{p}+\varepsilon)}\|\Delta_q g\|_{L^p}\|\nabla f\|_{L^\infty} \\ &\lesssim \|g\|_{L^p}\|\nabla f\|_{L^\infty}. \end{aligned}$$

Since

$$\Delta_j \sum_{q\in\mathbb{N}} \pi_1^q = \sum_{|j-q|\leq 4} \pi_1^q$$

then it follows

$$\|\pi_1\|_{C^\varepsilon} \lesssim \|g\|_{L^p}\|\nabla f\|_{L^\infty}. \tag{72}$$

Concerning the second term π_2^q, we follow the same steps of the preceding case

$$\begin{aligned} 2^{q\varepsilon}\|\pi_2^q\|_{L^\infty} &\lesssim 2^{q(-1+\varepsilon)}\|S_{q-1}g\|_{L^\infty}\|\nabla\Delta_q f\|_{L^\infty} \\ &\lesssim 2^{q(-1+\varepsilon+\frac{2}{p})}\|S_{q-1}g\|_{L^p}\|\nabla f\|_{L^\infty} \\ &\lesssim \|g\|_{L^p}\|\nabla f\|_{L^\infty}. \end{aligned}$$

Now we can conclude in a similar way to the first term π_1 that

$$\|\pi_2\|_{C^\varepsilon} \lesssim \|g\|_{L^p}\|\nabla f\|_{L^\infty}. \tag{73}$$

Let us now move to the third term π_3. By the definition of the remainder term we have

$$\begin{aligned} \|\Delta_q\pi_3\|_{L^\infty} &\lesssim \sum_{k\geq q-3} \|\big[\mathcal{R}_{ij},\Delta_k f\big]\tilde{\Delta}_k g\|_{L^\infty} + \big\|\big[\mathcal{R}_{ij},\Delta_{-1}f\big]\tilde{\Delta}_{-1}g\big\|_{L^\infty}\chi_{[-1,4]}(q) \\ &\lesssim \Big\{\sum_{k\geq q-3}\big\|\mathcal{R}_{ij}(\Delta_k f\,\tilde{\Delta}_k g)\big\|_{L^\infty} + \sum_{k\geq q-3}\big\|\Delta_k f(\mathcal{R}_{ij}\tilde{\Delta}_k g)\big\|_{L^\infty}\Big\} \\ &+ \big\|\big[\mathcal{R}_{ij},\Delta_{-1}f\big]\tilde{\Delta}_{-1}g\big\|_{L^\infty}\chi_{[-1,4]}(q) \\ &\triangleq \mathcal{I}_q + \mathcal{II}_q. \end{aligned}$$

By Bernstein inequality and the continuity of Riesz transforms over L^p we get

$$\begin{aligned} 2^{q\varepsilon}\mathcal{I}_q &\lesssim \|g\|_{L^p}2^{q\varepsilon}\sum_{k\geq q-3} 2^{k\frac{2}{p}}\|\Delta_k f\|_{L^\infty} \\ &\lesssim \|g\|_{L^p}\|\nabla f\|_{L^\infty}2^{q\varepsilon}\sum_{k\geq q-3} 2^{k(\frac{2}{p}-1)} \\ &\lesssim \|g\|_{L^p}\|\nabla f\|_{L^\infty}. \end{aligned} \tag{74}$$

For the low frequency term $\mathcal{II}_q$ we use once again Bernstein inequality combined with the continuity of Riesz transforms over L^p

$$\begin{aligned} \|\mathcal{II}_q\|_{L^p} &\lesssim \|\Delta_{-1}f\|_{L^\infty}\|\tilde{\Delta}_{-1}g\|_{L^p} \\ &\leq \|f\|_{L^\infty}\|g\|_{L^p}. \end{aligned}$$

Consequently we find

$$\|\pi_3\|_{C^\varepsilon} \lesssim \big(\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty}\big)\|g\|_{L^p}. \tag{75}$$

Therefore putting together (72), (73) and (75) yields

$$\big\|\big[\mathcal{R}_{ij}, f\big]g\big\|_{C^\varepsilon} \lesssim (\|f\|_{L^\infty} + \|\nabla f\|_{L^\infty})\|g\|_{L^p}.$$

This completes the proof of the commutator estimate.

□

Now, we introduce the following operator $\mathcal{L} := \partial_i \Delta^{-1}$ which is of convolution type and our aim is to establish a commutator estimate between this singular operator and the convection operator $v \cdot \nabla$.

Lemma 7.3. *Let $\varepsilon \in]0,1[$, $p, \in]1,\infty[$. Let $\rho : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function and v be a smooth divergence-free vector field on $\mathbb{R}^2$. Then*

$$\big\|\big[\mathcal{L}, v\cdot\nabla\big]\rho\big\|_{C^\varepsilon} \lesssim \|v\|_{C^\varepsilon}\|\rho\|_{L^\infty \cap L^p}.$$

Proof. The proof will be done in the spirit of the preceding one. From Bony's decomposition,

$$\begin{aligned}
\big[\mathcal{L}, v\cdot\nabla\big]\rho &= \sum_{q\in\mathbb{N}} \big[\mathcal{L}, S_{q-1}v\cdot\nabla\big]\Delta_q\rho + \sum_{q\in\mathbb{N}} \big[\mathcal{L}, \Delta_q v\cdot\nabla\big]S_{q-1}\rho + \sum_{q\geq -1} \big[\mathcal{L}, \Delta_q v\cdot\nabla\big]\tilde{\Delta}_q\rho \\
&\triangleq \sum_{q\in\mathbb{N}} \pi_1^q + \sum_{q\in\mathbb{N}} \pi_2^q + \sum_{q\geq -1} \pi_3^q \\
&\triangleq \pi_1 + \pi_2 + \pi_3.
\end{aligned}$$

To estimate the first term π_1^q we use its convolution structure,

$$\pi_1^q = h_q \star (S_{q-1}v\Delta_q\nabla\rho) - S_{q-1}v(h_q \star \nabla\Delta_q\rho),$$

where $\widehat{h}_q(\xi) = \frac{\xi_i}{|\xi|^2}\psi(2^{-q}\xi)$ and ψ is a smooth function supported in an annulus with center zero. Therefore $h_q(x) = 2^q h(2^q x)$ with $h \in \mathcal{S}$ and in view of Lemma 7.1 we get,

$$\begin{aligned}
\|\pi_1^q\|_{L^\infty} &\lesssim 2^{-2q}\|\nabla S_{q-1}v\|_{L^\infty}\|\Delta_q\nabla\rho\|_{L^\infty} \\
&\lesssim 2^{-q}\|\nabla S_{q-1}v\|_{L^\infty}\|\Delta_q\rho\|_{L^\infty}.
\end{aligned}$$

Hence we obtain since $\varepsilon < 1$

$$\begin{aligned}
2^{q\varepsilon}\|\pi_1^q\|_{L^\infty} &\lesssim 2^{q(-1+\varepsilon)}\|\rho\|_{L^\infty} \sum_{-1\leq j\leq q-2} \|\nabla\Delta_j v\|_{L^\infty} \\
&\lesssim 2^{q(-1+\varepsilon)}\|\rho\|_{L^\infty} \sum_{-1\leq j\leq q-2} 2^{j(1-\varepsilon)}\|v\|_{C^\varepsilon} \\
&\lesssim \|\rho\|_{L^\infty}\|v\|_{C^\varepsilon}.
\end{aligned}$$

Therefore we get

$$\|\pi_1\|_{C^\varepsilon} \lesssim \|\rho\|_{L^\infty}\|v\|_{C^\varepsilon}. \tag{76}$$

Concerning the second term π_2^q, we imitate the previous computations

$$\begin{aligned}
\|\pi_2^q\|_{L^\infty} &\lesssim 2^{-2q}\|\nabla\Delta_q v\|_{L^\infty}\|S_{q-1}\nabla\rho\|_{L^\infty} \\
&\lesssim \|\Delta_q v\|_{L^\infty}\|\rho\|_{L^\infty}.
\end{aligned}$$

Therefore we obtain

$$2^{q\varepsilon}\|\pi_2^q\|_{L^\infty} \lesssim \|v\|_{C^\varepsilon}\|\rho\|_{L^\infty}$$

and consequently

$$\|\pi_2\|_{C^\varepsilon} \lesssim \|\rho\|_{L^\infty}\|v\|_{C^\varepsilon}. \tag{77}$$

Let us now move to the third term π_3. By the definition of the remainder term we have

$$\begin{aligned}\pi_3 &= \sum_{q\geq -1}\mathcal{L}\mathrm{div}(\Delta_q v\,\tilde{\Delta}_q\rho)-\sum_{q\geq -1}\Delta_q v\cdot\nabla\mathcal{L}(\tilde{\Delta}_q\rho)\\ &\triangleq \pi_3^1-\pi_3^2.\end{aligned}$$

By Bernstein inequality we obtain for $j\in\mathbb{N}$

$$\begin{aligned}2^{j\varepsilon}\|\Delta_j\pi_3^1\|_{L^\infty} &\lesssim 2^{j\varepsilon}\sum_{q\geq j-4}\|\Delta_q v\|_{L^\infty}\|\tilde{\Delta}_q\rho\|_{L^\infty}\\ &\lesssim \|\rho\|_{L^\infty}\|v\|_{C^\varepsilon}\sum_{q\geq j-4}2^{(j-q)\varepsilon}\\ &\lesssim \|\rho\|_{L^\infty}\|v\|_{C^\varepsilon}.\end{aligned}$$

For the low frequency we use the continuity of Riesz transforms over L^p

$$\begin{aligned}\|\Delta_{-1}\pi_3^1\|_{L^\infty} &\lesssim \sum_{q\geq -1}\|\Delta_q v\|_{L^\infty}\|\tilde{\Delta}_q\rho\|_{L^p}\\ &\lesssim \|v\|_{C^\varepsilon}\|\rho\|_{L^p}.\end{aligned}$$

Thus we find

$$\|\pi_3^1\|_{C^\varepsilon} \lesssim \|v\|_{C^\varepsilon}\|\rho\|_{L^p\cap L^\infty}. \tag{78}$$

As regards the term π_3^2 we write

$$\begin{aligned}\pi_3^2 &= \sum_{q\geq 2}\Delta_q v\cdot\nabla\mathcal{L}(\tilde{\Delta}_q\rho)+\sum_{q=-1}^{1}\Delta_q v\cdot\nabla\mathcal{L}(\tilde{\Delta}_q\rho)\\ &\triangleq \pi_3^{2,1}+\pi_3^{2,2}.\end{aligned}$$

Since for $q\geq 2$ the Fourier transform of $\tilde{\Delta}_q\rho$ is supported in an annulus of size 2^q then

$$\begin{aligned}2^{j\varepsilon}\|\Delta_j\pi_3^{2,1}\|_{L^\infty} &\lesssim 2^{j\varepsilon}\sum_{q\geq j-4;q\geq 2}\|\Delta_q v\|_{L^\infty}\|\nabla\mathcal{L}(\tilde{\Delta}_q\rho)\|_{L^\infty}\\ &\lesssim \|\rho\|_{L^\infty}2^{j\varepsilon}\sum_{q\geq j-4}\|\Delta_q v\|_{L^\infty}\\ &\lesssim \|\rho\|_{L^\infty}\|v\|_{C^\varepsilon}\end{aligned}$$

For the term $\pi_3^{2,2}$ we get

$$\begin{aligned}\|\pi_3^{2,2}\|_{C^\varepsilon} &\lesssim \sum_{q=-1}^{1}\|\Delta_q v\|_{L^\infty}\|\nabla\mathcal{L}(\tilde{\Delta}_q\rho)\|_{L^p}\\ &\lesssim \|v\|_{C^\varepsilon}\|\rho\|_{L^p}.\end{aligned}$$

It follows that

$$\|\pi_3^2\|_{C^\varepsilon} \lesssim \|v\|_{C^\varepsilon}\|\rho\|_{L^p\cap L^\infty}. \tag{79}$$

Putting together (78) and (79) we find

$$\|\pi_3\|_{C^\varepsilon} \lesssim \|v\|_{C^\varepsilon}\|\rho\|_{L^p\cap L^\infty}.$$

Combining this estimate with (76) and (77) yields for any $p\in]1,\infty[$and $0<\varepsilon<1$,

$$\big\|\big[\mathcal{L},v\cdot\nabla\big]\rho\big\|_{C^\varepsilon}\lesssim\|v\|_{C^\varepsilon}\|\rho\|_{L^p\cap L^\infty}. \tag{80}$$

This completes the proof of the commutator estimate. $\square$

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